TRANSFERRING LOAD TO FLESH
PART II. ANALYSIS OF COMPRESSIVE STRESS[a]

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ABSTRACT

The normal stresses developed within a mass of flesh subject to various types of loading are examined theoretically. Using a simplified model of flesh possessing a definite modulus of elasticity and Poisson's ratio, static compressive stress values at various locations are determined. Formulas and numerous worked examples are given. The characteristics of load-transfer members including rigid blocks, thin covers, and membranes impinging on flesh are discussed in terms of compressive stress (quantitative) and shear stress (relative).

INTRODUCTION

The mating of a prosthetic device to a wearer implies the transfer of load at the interface between man and device. Occasionally it is possible to anchor the prosthesis directly to the bone structure of the wearer, as in dental work. More generally it is necessary that some combination of flesh and bone receive the device and sustain the load imparted by the device. Noting that the magnitude of such loads can be large (greater than the weight of a man in the case of a prosthetic leg) and that the attachment points are not designed by nature to sustain such loads, it is not surprising that cases of flesh failure occur (skin, trauma, lesions, abrasion). There is also dermatological evidence (1, 2) that the flesh in proximity to the edge of a load-bearing area generates an unusually large number of cysts as compared to areas remote from loading. Further, it is well known that local flesh loading influences edema formation and local blood circulation characteristics.

Murphy (3) (see Part I), concerned with stiffness gradient, shear concentration and their prosthetic implications, suggested that "frequent difficulties are due to excessive stress (and deformation) in the skin at the juncture with the stiffer appliance." That is, the abrupt transition

[a] Based on work performed under VA Contract V101(134)P-18.

between the rigid socket brim and more flexible stump is viewed as a stress raiser. The suggestion is made that stiffness of the appliance be so regulated as to diminish gradually as the edge is approached.

In view of the significance of prosthetic-transfer forces as concerns local trauma, it is surprising that the literature reflects but little knowledge of the flesh stresses arising from the application of such forces. What is desired is not the extreme case of automobile accidents (4) and high velocity falls (5) in which conditions of ultimate body failure are developed, but rather a series of solutions to prosaic (but usually repeated) cases, such as the stresses experienced in walking with a pebble, a nail, a grain of sand, or a wrinkled sock in one's shoe.

Practical questions to be considered include:

a. Does a relatively high stress exist in that flesh just proximal to the brim of a prosthetic socket?

b. If so, can the stress level be reduced by such means as a flexible socket or tapered socket?

c. Is a membrane (corset) superior in load transfer characteristics to a rigid member?

The clinician likely has opinions on these matters, backed by years of observation. However, adequate clinical experience with a new variable (soft liners, flexible sockets, etc.) takes many years to acquire and does not lend itself to performance prediction of yet untried elements. The use of theoretical mechanics in this area may be viewed as an attempt to rationalize the experience of the clinician and hopefully to extend his predictive powers in dealing with fresh situations. It is not intended that a theoretical approach be used as a substitute for clinical experience; it is hoped that ultimately a practical blend of the theoretical and the practical will emerge as a tool.

Our approach then, will be theoretical, based on stress analysis techniques. However, the practical questions posed above do not lend themselves to solution by exclusive use of the classic "theory of elasticity" procedures. Consider, for example, the problem illustrated in Figure 1. A hard block is pressed into a softer but elastic medium. According to classic elasticity theory (6) the pressure distribution in the soft medium is given by $q=\frac{P}{\pi\sqrt{\left(\frac{l}{2}\right)^2-x^2}}$ where q is the local pressure at coordinate x resultant from the application of load P through a hard block of half-width $\frac{l}{2}$. While the given solution may possess qualitative merit, it does not properly describe the pressure near the corners of the block. Indeed the solution "blows up" when $x=\frac{l}{2}$ pointing to a dis-

continuity of infinite pressure that does not represent physical truth. In reality, the deflections near the corner may be so great as to invalidate the small deflection assumption implicit in elasticity solutions, or sufficient plastic flow will occur in the corner region to permit finite stresses. Therefore, traditional elasticity theory is inadequate for our needs.

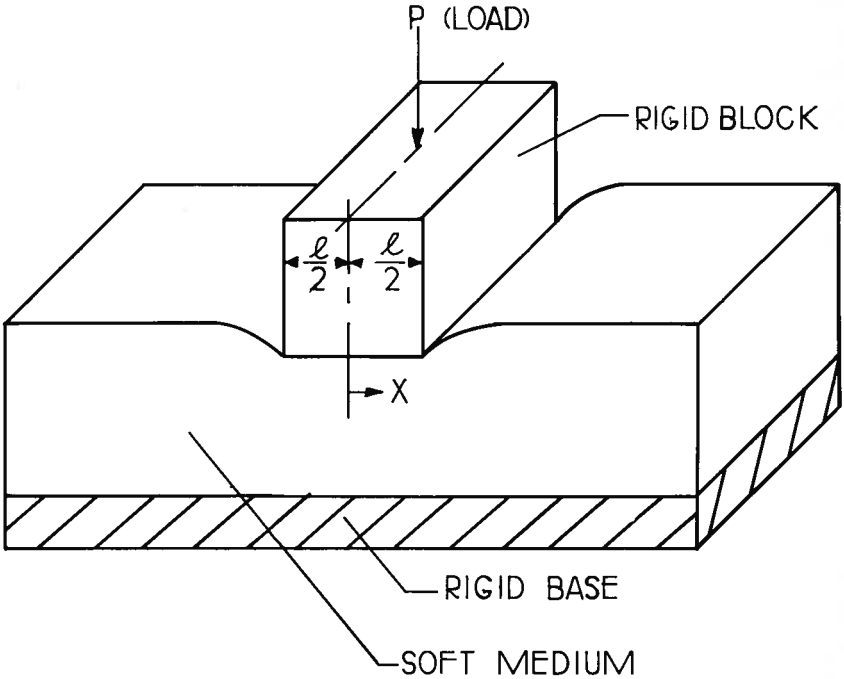


FIGURE 1.— Sketch and nomenclature employed in classic stress solution by Sadowsky.

A means of properly handling such problems (the mating of hard and soft components) has been developed in recent years, principally by Russian workers. A practical interest in the stress field developed under foundations of buildings placed on marginal soil has resulted in a complete body of theory in which the traditional elasticity theory has been combined with a variational approach and a realistic model of material limitations.

The similarity of the "foundation" model to the situation existent in proximity of a prosthesis to its neighboring flesh seems reasonable. In what follows we have applied the Russian work as compiled by Vlasov and Leontev (7) to the problem of load transfer to flesh. No claim for originality of theory is made; the task has been one of application rather than origination.

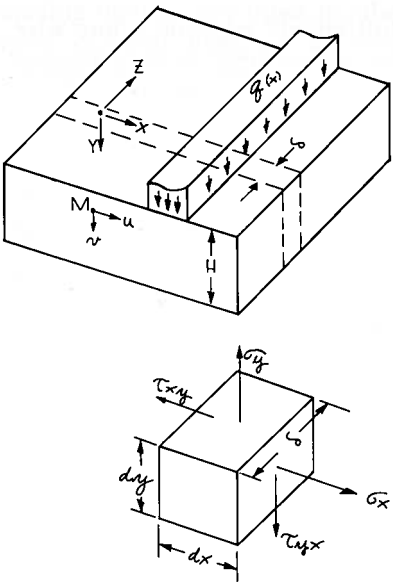
The application of tools and techniques found successful in one area

to another is not without danger. The quality of this work can be no better than the quality of the underlying assumptions, and such assumptions are frequently gross. Thus, we shall treat flesh as though it possessed a fixed modulus of elasticity and a fixed Poisson's ratio (of lateral contraction to longitudinal extension, causing slimming of a stretched material) despite evidence (8) that both are dependent functions of the stress level. We shall also take flesh to be homogeneous and isotropic although it is well known that flesh is a complex multilayered material whose properties often vary with direction. Further, we shall ignore the known visco-elastic properties of flesh (and the fluids it contains) and assume all stress states to be time-independent. No rationale is offered for these simplifying assumptions beyond the necessity, in any initial effort, to reduce the problem scope to something manageable. Hopefully as work continues, it will be possible to introduce increasingly sophisticated flesh models so as to arrive at more precise results. For the moment it may be said only that the following results, based on necessarily crude assumptions, must be viewed as a correspondingly crude approximation to a complex reality. We shall first study some simple cases to see if they match our intuitionally derived notions and clinical experience, then move onward.

METHOD OF PROCEDURE

The essential derivations are given in great detail by Vlasov and Leontev (7). While this work is readily available in an English translation, it is difficult to follow owing to an encyclopedic style, numerous typographical errors, and translational problems. Consequently, a sample derivation has been worked fully in the appendix to this paper. Also listed in the appendix are the answers (i.e., compressive stress in relation to loading) to a number of cases of interest. Given immediately below is a short summary of the basic procedure plus an abbreviated list of nomenclature containing key symbols. See Figure 2 for a depiction of the coordinate system and certain physical factors.

We are seeking the compressive stress experienced in a uniform fleshy mass squeezed between an external load and a bony foundation. We shall employ the method of displacements, in which the unknowns are the displacements u (x,y) and v (x,y) of a certain point M (x,y) contained within a mass of elastic flesh of thickness H. All problems considered will be two-dimensional, i.e., extending without end along the Z axis. Therefore, we need consider only a typical slice of width δ (see Fig. 2, upper sketch) in our work. The flesh is considered to possess a modulus of elasticity E_s, a Poisson's ratio ν_s, normal stresses σ_x (longitudinal) and σ_y (vertical), shear stresses τ_{xy} and τ_{yx}, and strain components ϵ_{xx}, ϵ_{yy}, ϵ_{xy}.



COORDINATE SYSTEM EMPLOYED IN THIS WORK

FIGURE 2.—Upper sketch shows loads q(x) on flesh of depth H containing point M located at x,y,z (located arbitrarily with respect to the loaded area). Point M will experience deflections u(x,y) and v(x,y). A slice of width δ, taken along the dashed lines, is shown (lower sketch) with acting normal stresses σ_x and σ_y and the shear stresses τ_{yx} and τ_{xy}. For clarity the forces and shears on the opposite faces are omitted. In accordance with the assumption of two-dimensional loading, there are no forces and no shears in the z direction.

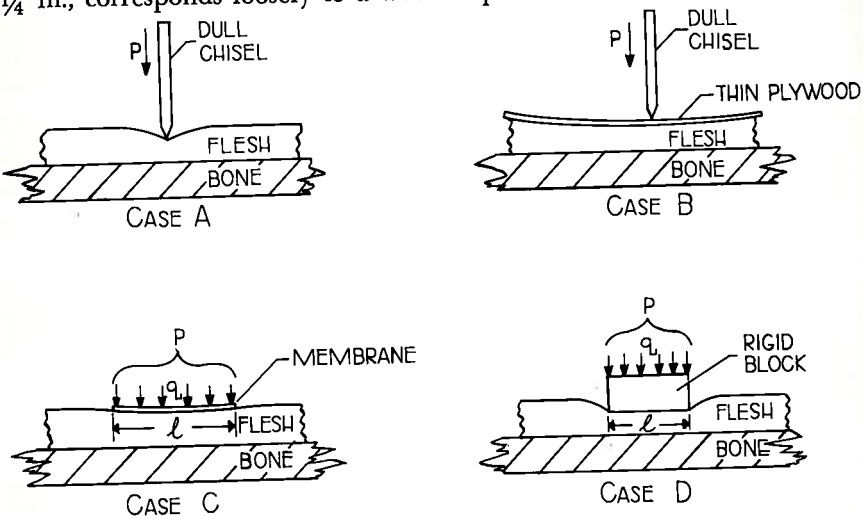
We proceed by first using the conventional interrelationships between the stresses and strains taken from plane elasticity theory to establish useful identities. Then the unknown displacements, by the method of separation of variables, are set equal to a series summation of the products of x and y dependent functions. The x dependent functions are determined by strain energy methods. The y dependent functions are those establishing stress (or strain) attenuation with depth and are sensitive to the material involved. The latter values should be established through empirical testing. Lacking appropriate values from tests on flesh, we employ values found effective in testing soft soils. By combining all the above concepts, a general expression for compressive stress under load is produced. Variations of load type are treated as special cases, and appropriate expressions are derived. Finally, insertion of test values (on cat skin) for Poisson's ratio and modulus of elasticity permit the working of numerous examples, providing a sense of resulting trends.

Standard hyperbolic function tables plus slide-rule calculations lead to the final numerical results. Stress units are in pounds per square inch (p.s.i.). After initial reconnaissance to determine the crucial questions, transfer to computer techniques may prove useful, or the appropriate paths for clinical trials may become clear.

RESULTS

The special cases solved for compressive stress are shown in Figure 3.

Case A is that due to loading of a "dull chisel" edge, or a fine wire, impinging directly on flesh supported by a bone base. It is understood that the chisel is not so sharp as to slice through the flesh. In Case B a thin and moderately flexible piece of plywood has been inserted between the dull chisel and the flesh. All other factors are identical. Two thicknesses of plywood have been considered. The thinner, 1/16 in., is roughly equal to a tongue depressor in its characteristics. The thicker, and stiffer, ¼ in., corresponds loosely to a wooden pencil.



TYPES OF LOADING STUDIED

FIGURE 3.—In each case a layer of flesh of thickness H is supported by rigid bone. The loads are applied in the following manner:

Case A—Dull chisel impinges directly on the flesh with load P.

Case B—Identical to A with the exception of a very wide but thin sheet of plywood inserted between chisel and flesh.

Case C—Membrane (or corset) incapable of taking a bending load and quite soft under tension is forced against the flesh with unit loading q over width *l*. The total load equals P.

Case D—A rigid block of width *l* is forced into the flesh under unit loading q. The total load equals P.

In Case C a membrane is pushed against the bone supported flesh, with uniform pressure. A tight bandage, an elastic sock, or a corset (lacking stays) are examples of this type of loading. Case D loading is that due to an absolutely rigid block pressing against the flesh. A brace support or a prosthetic brim typifies this type of load.

A typical series of results is given in Figure 4 for Cases A and B. The upper and lower figures at the left end give the stresses for flesh thicknesses of 1 and 3 in., Case A. The center depicts Case B results with a thin (1/16 in.) plywood cover, the right end, Case B with a thicker and stiffer (¼ in.) cover. The stresses in the figure (and suc-

ceeding figures) are those numbers written at a given locale times the value of load. Thus, for a given load, all stress figures in this work are directly comparable. For example, the stress immediately under the load in the Case A condition, 1 in. flesh thickness (upper left, Fig. 4) is 2.12 times the applied load, or 212 percent of the load. Applying a thin plywood cover (upper middle, Fig. 4) cuts the local stress almost in half to 1.11. Increasing the cover thickness further (upper right, Fig. 4) and hence stiffening the plywood plate markedly reduces the local stress by a factor of 10 below that encountered with the "dull chisel."

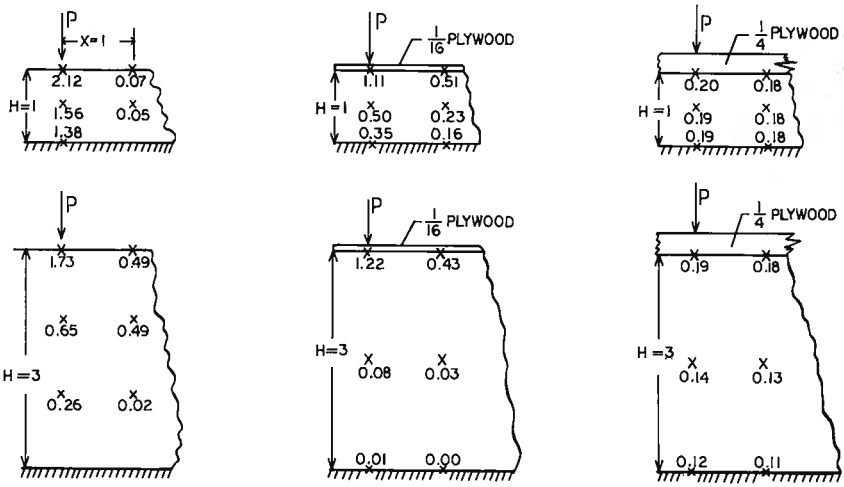


FIGURE 4.—Normal stress within flesh; Cases A and B. Poisson's ratio of flesh taken as ⅓. Stress at a locale equals numerical value shown, times P.

The cover acts to distribute stresses laterally, thus reducing local peak values. "Smoothing" is almost complete in the case of the thick plywood; the results show an almost uniform stress pattern. It follows that relative to 1 in. of flesh, a plywood thickness of ¼ in. approaches the infinitely rigid condition.

Increasing the flesh thickness (lower set, Fig. 4) acts to distribute stresses laterally (lower left), somewhat like the plywood cover. It also promotes greater attenuation with depth. That is, stress values near the rigid bony base are decidedly lower for a greater flesh thickness. Thus, with all other factors identical, the greater flesh thickness is better able to sustain a load. Application of covers is useful in further smoothing and lowering peak values (lower middle and right).

In calculating the results given in Figure 4, a single value of Poisson's ratio has been used (0.33, comparable to some common engineering materials). Here Poisson's ratio is defined as the ratio of lateral contraction

to the longitudinal extension. Studies of cat skin (8) indicate Poisson's ratio to be a function of the load applied; the higher the load, the higher the Poisson's ratio. Therefore, it is of interest to examine the effect of change in Poisson's ratio to see how serious errors in its estimation might be.

In Figure 5, Case A of thick flesh under "chisel" loading is reexamined as a function of Poisson's ratio. As the Poisson's ratio increases, the effect is to reduce stress values remote from the loading axis. Stress along the loading axis is not influenced. While the results are few, it would appear that under conditions of increasing load, leading to higher Poisson's ratio, the results given here are conservative (possibly too high) at points remote from the loading site, and that compression stresses directly under the "dull chisel" are not affected by reasonable ranges of Poisson's ratio.

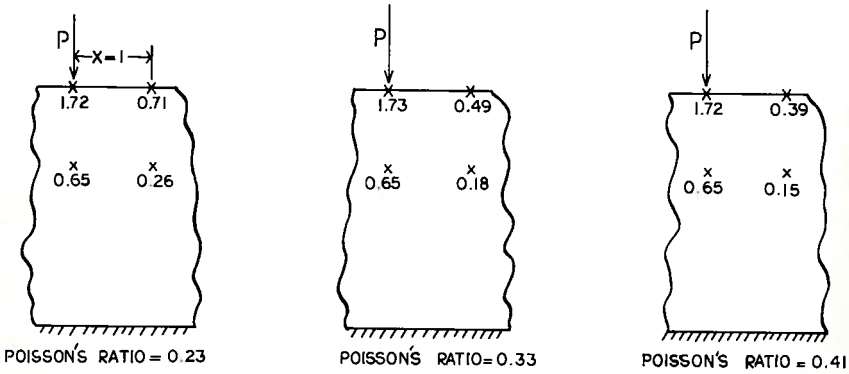


FIGURE 5.—Normal stress within flesh *versus* Poisson's ratio. Stress at a locale equals numerical value shown times P. Flesh thickness H = 3. Case A Load.

A different means of presentation of stress results is employed in the remaining cases (C and D). See Figures 6 through 9. The stress at a given level (surface or inner) is plotted as a function of the distance from the centerline of the load. Complete definitions of the loading types are given in Figure 3. Various widths are employed; results are given of calculations for loading surface 2 in., 6 in., and 20 in. wide. Both thick (Fig. 6 and 8) and thin (Fig. 7 and 9) sections of flesh are considered. As before, the thick flesh section is 3 in. deep and the thin, 1 in.

As one would expect for these two cases (C and D), peak stresses are produced under the narrowest load, regardless of flesh thickness or type of loading member. The inner surface is always loaded, especially with narrow membrane or block, to a lesser degree than the surface layer,

although in the case of the widest rigid block, the stress levels at the surface and on a layer 1 in. below are essentially the same.

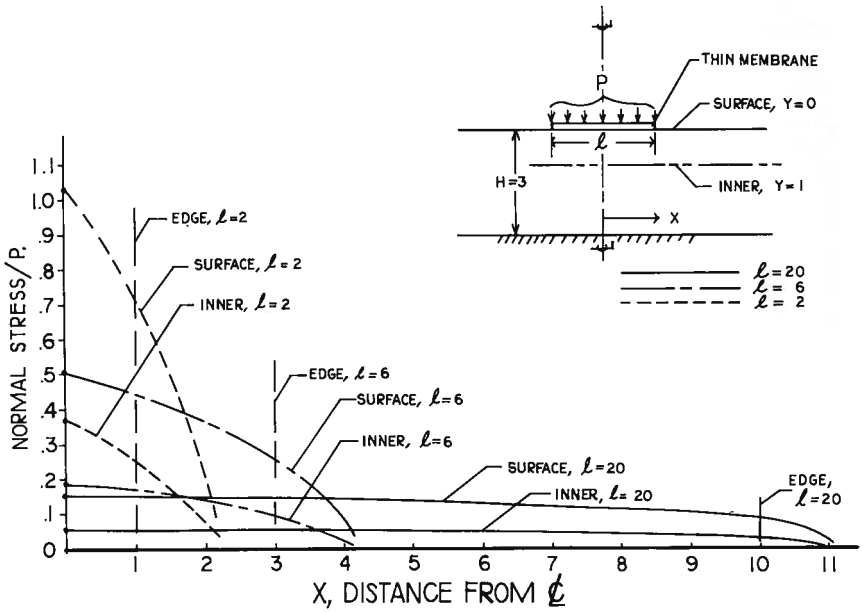


FIGURE 6.—Results, Case C, flesh thickness = 3.

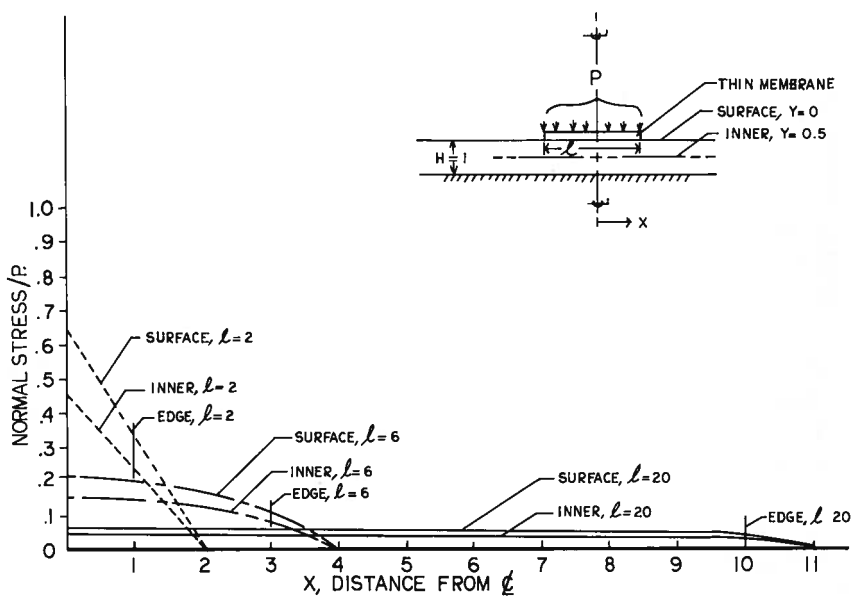


FIGURE 7.—Results, Case C, flesh thickness = 1.

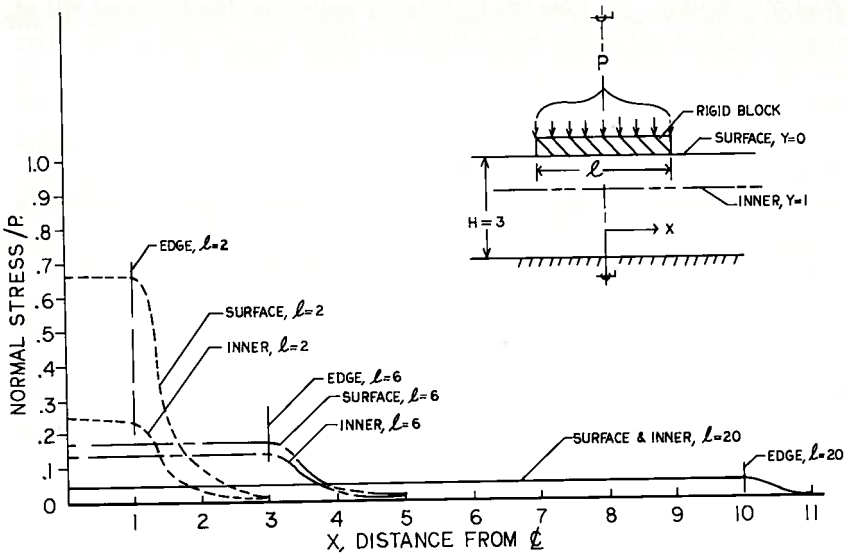


FIGURE 8.—Results, Case D, flesh thickness = 3.

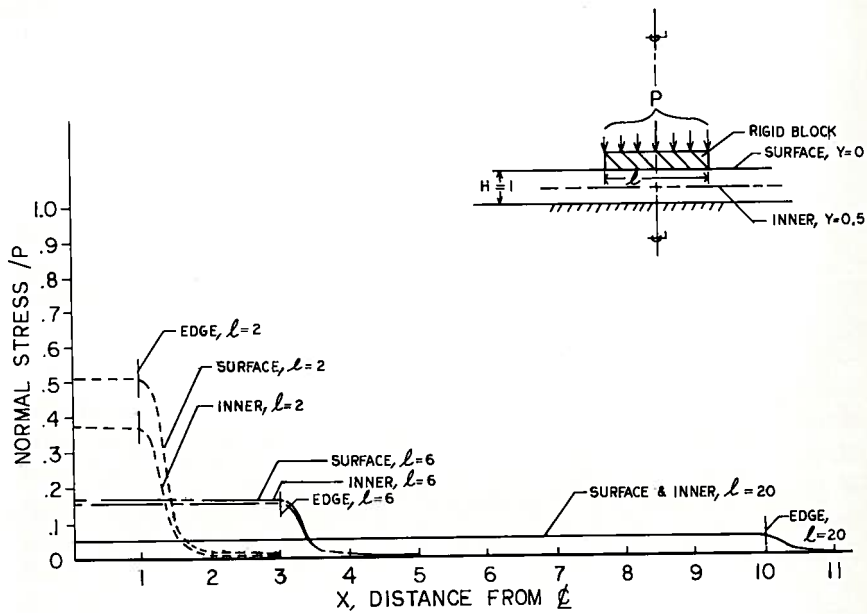


FIGURE 9.—Results, Case D, flesh thickness = 1.

The effect of flesh thickness on stress magnitude is not simple; no clear relationship between the two is apparent. More thicknesses should be studied, especially a simulation of thin flesh over a bony prominence.

However, it can be said that a smaller flesh thickness appears to produce a more uniform distribution (in a top to bottom sense) within the flesh.

Compressive stresses in that flesh beyond the direct impress of the loading member fall off to zero in slightly more than 1 in. Nonetheless, such stresses may be large. For example in the case of the narrow membrane a stress level equal to 60 percent of the peak is developed *beyond* the membrane (see Fig. 6 curve labeled "surface, $l=2$"). The effects of the sharp gradients calculated for some cases are currently being studied.

DISCUSSION

Consider the matter of credibility. This work is purely theoretical and lacks any experimental proof. The theory is drawn partly from the classic theory of elasticity, partly from well-known work-energy relationships, and partly from empirical testing of soft media. While the elasticity and work-energy contributions are well proven, the empirical relationships, which are not based on flesh tests, warrant a skeptical approach. Hopefully the physiologists will one day, through direct animal testing with implanted gages, supply the appropriate empirical relationships (functions φ and ψ). In the meantime this work might be viewed as less than the whole truth—yet considerably more than a guess.

The results in general confirm intuitive expectations. Thus "dull chisel" loading (Case A) is the most severe; even a thin cover plate (Case B with 1/16-in. thickness) greatly reduces local compressive stress. As the cover plate becomes rigid (Case D) the main effect is to even out the stresses in that flesh under the cover. Compressive stresses in that flesh beyond the direct impress of load rapidly decay to negligible values in a distance of roughly 1¼ in. or less—though there are stresses and stress gradients well beyond the edge. All of these results appear intuitively reasonable.

The membrane loading (Case C, Fig. 6) in the narrow width does offer a surprise; the maximum stress developed on center approaches that of the blunt chisel condition (Case A, Fig. 4, lower left). It would appear that loading to flesh through a narrow corset arrangement is not optimal from the viewpoint of maximum compressive stress. However, the corset (membrane) compressive stress gradient with respect to X is smaller and more nearly uniform than comparable rigid loading cases. This may be seen by inspecting Figures 6–9 and comparing the slope of the curves. The slope (gradient with respect to X) is a rough measure of shear stress. This follows from equations (1) and (2). Therefore, it may be hypothesized that the corset loading may cause lower and more uniform shear stresses within the flesh than corresponding rigid loading members. Current work is expanding the study of stress gradients just beyond the edge of a membrane or loading structure.

As concerns those questions raised in the introduction, it may be said that:

a. A large stress exists proximal to the brim of a prosthetic socket, or especially on either side of a narrow cuff of an orthotic device, if we assume a fleshy mass compressed between the bone and the brim. Appreciable compressive stress will be experienced to a distance of 1 in. above the socket brim.

b. The stress level at a desired location can be lowered through a process of leveling or smoothing controlled by the local stiffness of the prosthetic structure.

c. Corsets (membranes) may cause a larger maximum compressive stress than a comparable rigid member. However, the shear stress in the stress caused by corset loading appears generally to be lower and more nearly uniform.

In this work we have concentrated largely on compressive stress with but a few qualitative comments on shear stress. The choice of emphasis is arbitrary; no comment is implied with respect to the relative significance of compressive stress and shear stress as causative factors in trauma, and no attention has been given to possible stresses from loads on or displacements of the socket or cuff parallel to the flesh surface ("pumping").

ACKNOWLEDGEMENTS

This work was supported by the R & D Division of the Prosthetic and Sensory Aids Service, Veterans Administration. Dr. Eugene F. Murphy of that organization has contributed many useful suggestions in the course of lively discussions.

APPENDIX

This appendix contains a full derivation of the compressive stresses experienced in Case A, plus the final expressions for stress in Cases B, C, and D. A portion of the nomenclature is given in the Method of Procedure and Figure 2.

We are considering a two-dimensional or plane stress case. Useful relationships, taken from the theory of elasticity, relate the stresses and strains as follows:

$$\left.\begin{aligned} \sigma_x &= \frac{E_o}{1-\nu_o^2}(\epsilon_{xx}+\nu_o\epsilon_{yy}) \\ \sigma_y &= \frac{E_o}{1-\nu_o^2}(\epsilon_{yy}+\nu_o\epsilon_{xx}) \\ \tau_{xy} &= \tau_{yx} = \frac{E_o}{2(1+\nu_o)}\epsilon_{xy} \end{aligned}\right\} \quad (1)$$

$$\left.\begin{aligned} &\text{where} \quad E_o=\frac{E_S}{1-\nu_S^2},\ \nu_o=\frac{\nu_S}{1-\nu_S} \\ &\text{and} \quad \epsilon_{xx}=\frac{\partial u}{\partial x},\ \epsilon_{yy}=\frac{\partial v}{\partial y},\ \epsilon_{xy}=\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x} \end{aligned}\right\} \quad (2)$$

We are searching for the displacements u (x,y) and v (x,y) which may be expanded in a finite series as:

$$\left.\begin{aligned} u(x, y) &= \sum_{i=1}^{m} U_i(x)\,\varphi_i(y) \quad (i=1, 2, 3, \ldots\ldots m) \\ v(x, y) &= \sum_{k=1}^{n} V_k(x)\,\psi_k(y) \quad (k=1, 2, 3 \ldots\ldots n) \end{aligned}\right\} \quad (3)$$

That is, we assume that the displacements can be approximated by a finite number of terms of separate functions x and y. As a matter of convenience, the U and V terms, which are functions of x only, may be taken as the deflection at the surface of the flesh. Unitwise, the U and V terms may be taken as deflection units, say inches. The distributions of the longitudinal and vertical displacements over the section X=const. are given respectively by the functions φ and ψ. As a matter of convenience, these latter functions will be taken as dimensionless. Further, their values will be based on empirical evidence obtained through testing various soft materials (largely soil) and are offered without further proof. Thus the functions φ and ψ are assumed to be known and the functions U and V to be unknown.

To determine U and V, we shall use strain energy methods and in particular, the principle of virtual displacements in which equilibrium is obtained by setting to zero the total work of all internal and external forces acting on an elementary strip experiencing a virtual displacement.

In accordance with equation (3), the virtual longitudinal displacement of an elementary strip dx is $\bar{u}=\varphi_j(y)$ for a unit virtual displacement $U_j=1$ where j may take on m different values. If we take the rate of change of normal stress with x to be $\frac{\partial \sigma_x}{\partial x}$ the work done over the virtual displacement φ_j is equal to $\int \frac{\partial \sigma_x}{\partial x}\varphi_j \delta dy$. Continuing in a similar manner, and considering all external and internal forces, the total work done is:

$$\left.\begin{aligned} &\int \frac{\partial \sigma_x}{\partial x}\varphi_j \delta dy - \int \tau_{xy}\varphi_j' \delta dy + \int p\,(x, y)\,\varphi_j dy = o \quad (j=1, 2, 3, \ldots\ldots m) \\ &\int \frac{\partial \tau_{yx}}{\partial_x}\psi_h \delta dy - \int \sigma_y \psi_h' \delta dy + \int q\,(x, y)\,\psi_h dy = o \quad (h=1, 2, 3 \ldots\ldots n) \end{aligned}\right\} \quad (4)$$

where the prime terms represent the first derivative with respect to the dependent variable. Here the top equation (4) describes longitudinal (x) work and the bottom equation (4) describes transverse (y) work. In each equation the first term represents the work of the external forces, the second term represents the work of the internal forces, and the last team corresponds to the virtual work of the distributed loads of intensity p in the x direction and q in the y direction.

Combining equations (1), (2), and (3) we have

$$\left.\begin{aligned}
\sigma_x &= \frac{E_o}{1-\nu_o^2}\left[\sum_{i=1}^{m} U_i'\varphi_i + \nu_o \sum_{k=1}^{n} V_k\psi_k'\right] \\
\sigma_y &= \frac{E_o}{1-\nu_o^2}\left[\sum_{k=1}^{n} V_k\psi_k' + \nu_o \sum_{i=1}^{m} U_i'\varphi_i\right] \\
\tau_{xy} = \tau_{yx} &= \frac{E_o}{2(1+\nu_o)}\left[\sum_{i=1}^{m} U_i\varphi_i' + \sum_{k=1}^{n} V_k'\psi_k\right]
\end{aligned}\right\} \quad (5)$$

The substitution of (5) in (4) leads to the following system of (m+n) ordinary differential equations in U_i and V_k

$$\left.\begin{aligned}
&\sum_{i=1}^{m} a_{ji}U_i'' - \frac{1-\nu_o}{2}\sum_{i=1}^{m} b_{ji}U_i + \sum_{k=1}^{n}\left(\nu_o t_{jk} - \frac{1-\nu_o}{2}C_{jk}\right)V_k' \\
&\qquad + \frac{1-\nu_o^2}{E_o}p_j = 0 \qquad (j=1, 2, 3, \ldots\ldots m) \\
&-\sum_{i=1}^{m}\left(\nu_o t_{hi} - \frac{1-\nu_o}{2}C_{hi}\right)U_i' + \frac{1-\nu_o}{2}\sum_{k=1}^{n} r_{hk}V_k'' - \sum_{k=1}^{n} S_{hk}V_k \\
&\qquad + \frac{1-\nu_o^2}{E_o}q_h = o \qquad (h=1, 2, 3, \ldots\ldots n)
\end{aligned}\right\} \quad (6)$$

in which the coefficients are defined as below

$$\left.\begin{aligned}
a_{ji} &= \int \varphi_j\varphi_i \delta dy & r_{hk} &= \int \psi_h\psi_k \delta dy \\
b_{ji} &= \int \varphi_j'\varphi_i' \delta dy & s_{hk} &= \int \psi_h'\psi_k' \delta dy \\
c_{jk} &= \int \varphi_j'\psi_k \delta dy & c_{hi} &= \int \psi_h\varphi_i' \delta dy \\
t_{jk} &= \int \varphi_j\psi_k' \delta dy & t_{hi} &= \psi_h'\varphi_k \delta dy
\end{aligned}\right\} \quad (7)$$

and the load terms may be obtained from the given loading p (x, y) and q (x, y) through

$$p_j = \int p(x, y)\,\varphi_j dy \qquad q_h = \int q(x, y)\,\psi_h dy \qquad (8)$$

Equations (6), (7) and (8) represent the general solution for de-

flections U and V. Substituting the deflection expression into equation (5) will yield the stress value of interest.

Let us apply such a procedure in the case of a concentrated vertical force, specifically an infinitely wide (2 dimensional) chisel point impinging on flesh. The flesh is supported by a bone base which is assumed infinitely rigid. It is assumed that the chisel is dull and does not slice through the flesh. The question to be solved is the state of compressive stress within the flesh and in the vicinity of the load.

Assume that longitudinal displacements are small as compared to the vertical. (The chisel is vertical.) Then:

$$u(x, y) = o$$

$$v(x, y) = \sum_{k=1}^{n} V_k(x)\,\psi_k(y) = V_1(x)\,\psi_1(y) \tag{9}$$

where the displacements occur as a result of surface loads only. Equation (6) becomes

$$\frac{1-\nu_o}{2}\sum_{k=1}^{n} r_{hk}V_k'' - \sum_{k=1}^{n} S_{hk}V_k + \frac{1-\nu_o^2}{E_o}q_h = 0 \tag{10}$$

Again noting the single load source (10), may be written as

$$\frac{1-\nu_o}{2}r_{11}V_1'' - S_{11}V_1 + \frac{1-\nu_o^2}{E_o}q_1 = 0 \tag{11}$$

with coefficients

$$r_{11} = \int_o^H \psi_1^2\delta dy \qquad S_{11} = \int_o^H (\psi_1')^2\delta dy \tag{12}$$

The deflection equation (11) has been reduced to a homogeneous differential equation. After some manipulation, it may be written as

$$2tV_1'' - dV_1 = 0 \tag{13}$$

$$\text{where } t = \frac{E_o r_{11}}{4(1+\nu_o)} \qquad d = \frac{E_o S_{11}}{1-\nu_o^2} \tag{14}$$

The solution to (13) is known to be (Ref. 9)

$$V_1 = C_1 e^{-\alpha x} + C_2 e^{\alpha x} \tag{15}$$

$$\text{where } \alpha = \sqrt{\frac{d}{2t}} \tag{16}$$

Introducing $\psi_1(y) = \dfrac{\text{Sinh }(H-y)}{\text{Sinh } H}$ on a pragmatic basis, by (17)

the analogy of flesh to or near a socket to soil under or near a founda-

tion as previously explained, and inserting boundary conditions (as $x \rightarrow \infty$, $V_1 \rightarrow 0$) the final expression for the vertical deflection v can be shown to be

$$v\ (x, y) = \frac{3\,(1-\nu_o^2\ )}{\sqrt{6\,(1-\nu_o)}}\ \frac{Pe^{-\alpha x}}{E_o}\ \frac{\text{Sinh}\ (H-y)}{\text{Sinh}\ H}\ \frac{1}{\psi_t \psi_a} \tag{18}$$

where

$$\left.\begin{aligned} \alpha &= \frac{\sqrt{6\,(1-\nu_o)}}{(1-\nu_o)}\ \psi_a \\ \psi_t &= \frac{3}{2H}\ \frac{\text{Sinh H Cosh H}-H}{\text{Sinh}^2\ H} \\ \psi_a &= H\sqrt{\frac{\text{Sinh H Cosh H}+H}{3\ (\text{Sinh H Cosh H}-H)}} \end{aligned}\right\} \tag{19}$$

Substituting (15) through (19) into (5) leads to the final relationship:

$$\sigma_y = -\frac{3PE_s}{E_o\delta}\ \frac{1}{\sqrt{6\,(1-\nu_o)}}\ \frac{e^{-\alpha x}\ \text{Cosh}\ (H-y)}{\psi_t \psi_a\ \text{Sinh}\ H} \tag{20}$$

Equation (20) represents the normal stress (compression) experienced at a horizontal face within a mass of flesh squeezed between a dull chisel and a rigid bone. See Figure 3, Case A. Equation (20) readily accepts varied values of X and Y, so that a stress field can be constructed in the proximity of the loaded edge. Values of flesh stiffness and Poisson's ratio may also be modified as desired.

A determination of the normal stress has been made in a number of other cases, shown in Figure 3. The stress derivation is similar to that given above, with the difference of some added complexity due to the distribution of load. The resulting stresses are given by the following expressions:

Case B—Dull chisel impinging on flesh covered with thin plywood

$$\sigma_y = \frac{-P}{2\bar{\alpha}\bar{\beta}S^2}\left[\bar{\alpha}F_1\,(\eta) + \bar{\beta}F_2\,(\eta)\right]\left[\frac{1}{L\ \text{Sinh}\ \frac{H}{L}}\right]\left[\text{Cosh}\ \frac{H-y}{L}\right] \tag{21}$$

where $L = \sqrt[3]{\dfrac{2EJ\ (1-\nu_o^2)}{E_o\delta}}$ Elastic characteristic plywood

$J = \dfrac{\delta h^3}{12\ (1-\mu^2)}$ Plywood moment of inertia

h = thickness plywood

μ = Poisson's ratio of plywood

E = Modulus of elasticity of plywood

$$\bar{\alpha}=\sqrt{\frac{s^2+r^2}{2}}$$

$$s^4=2\frac{L}{H}\Psi_k$$

$$\Psi_k=\frac{H}{2L}\left(\frac{\text{Sinh}\frac{H}{L}\,\text{Cosh}\frac{H}{L}+\frac{H}{L}}{\text{Sinh}^2\frac{H}{L}}\right)$$

$$r^2=\frac{2\,(1-\nu_0)}{12}\frac{H}{L}\Psi_t$$

$$\Psi_t=\frac{3L}{2H}\left(\frac{\text{Sinh}\frac{H}{L}\,\text{Cosh}\frac{H}{L}-\frac{H}{L}}{\text{Sinh}^2\frac{H}{L}}\right)$$

$$\bar{\beta}=\sqrt{\frac{s^2-r^2}{2}}$$

$$F_1(\eta)=e^{-\bar{\alpha}\eta}\,\text{Sin}\,\bar{\beta}\eta$$

$$F_2(\eta)=e^{-\bar{\alpha}\eta}\,\text{Cos}\,\bar{\beta}\eta$$

$$\eta=\frac{x}{L}$$

Case C—Thin membrane impinging, with uniform pressure, on flesh.

$$\sigma_y=\text{Cosh}\ (H-y)\ \frac{V_1(x)}{(1-\nu_0^2)\ \text{Sinh}\ H} \tag{22}$$

where

$$V_1(x)=\frac{q}{2k}\left[2-e^{-\alpha(x-a)}-e^{\alpha(x-b)}\right]\ \text{under membrane}$$

$$V_1(x)=\frac{q}{2k}\left[e^{\alpha(x-b)}-e^{\alpha(x-a)}\right]\ \text{to left of membrane}$$

$$V_1(x)=\frac{q}{2k}\left[e^{\alpha(b-x)}-e^{\alpha(a-x)}\right]\ \text{to right of membrane}$$

a = distance from origin to left end membrane

x = distance from origin to point within flesh

b = distance from origin to right end membrane

$$\alpha=\sqrt{\frac{k}{2t}}$$

$$k=\frac{E_o\delta}{H(1-\nu_o^2)}\psi_k$$

$$t=\frac{E_o\delta H}{12(1+\nu_o)}\psi_t$$

$$\psi_k=\frac{H}{2}\frac{\text{Sinh H Cosh H}+H}{\text{Sinh}^2 H}$$

$$\psi_t=\frac{3}{2H}\frac{\text{Sinh H Cosh H}-H}{\text{Sinh}^2 H}$$

Case D—Rigid block impinging with uniform pressure (i.e., no tilting moment) on flesh

$$\sigma_y=\frac{E_oC_oe^{\alpha(x+l)}\,\text{Cosh}\frac{H-y}{l}}{(l-\nu_o^2)\,2\,(kl+2\alpha t)\,l\,\sinh\frac{H}{l}} \tag{23}$$

where

l=width of block

$$\alpha=\sqrt{\frac{k}{2t}}$$

$$k=\frac{E_o\delta}{H(1-\nu_o^2)}\left[\frac{H}{2l}\left(\frac{\text{Sinh}\frac{H}{l}\text{Cosh}\frac{H}{l}+\frac{H}{l}}{\text{Sinh}^2\frac{H}{l}}\right)\right]$$

$$t=\frac{E_o\delta H}{12(1+\nu_o)}\left[\frac{3l}{2H}\left(\frac{\text{Sinh}\frac{H}{l}\text{Cosh}\frac{H}{l}-\frac{H}{l}}{\text{Sinh}^2\frac{H}{l}}\right)\right]$$

$$\alpha=\frac{\sqrt{6}}{\sqrt{(1-\nu_o)}}\left[\frac{1}{l}\sqrt{\frac{\text{Sinh}\frac{H}{l}\text{Cosh}\frac{H}{l}+\frac{H}{l}}{3\,\text{Sinh}\frac{H}{l}\text{Cosh}\frac{H}{l}-\frac{H}{l}}}\right]$$

$$C_o=\frac{P}{2\,(kl+2\alpha t)}$$

C_o is the deflection of the block pressing into the flesh, whereas α, k, t are composite mathematical terms with no obvious physical meaning.

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