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A theory of wheelchair wheelie performance*

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Abstract—The results of this analytical study of wheelchair wheelie performance can be summarized into two wheelchair design equations, or rules of thumb, as developed in the paper. The equation containing the significant parameters involved in popping a wheelie for curb climbing is:

$$f_h = 0.8 \text{ mg } \theta_{c.g.} \qquad [A]$$

where f_h is handrim force, m is the mass of the wheelchair + user less rear wheels, g is acceleration of gravity (9.807 m/s^2), and $\theta_{c.g.}$ is "c.g. angle," i.e., the angle between the vertical through the rear axle and a line connecting the rear axle and the system center-of-gravity. Equation [A] shows that reducing the mass and/or the c.g. angle will make it easier to pop a wheelie. The c.g. angle is reduced by moving the rear axle position forward on the wheelchair. Wheelie balance is the other aspect of performance considered; where the user balances the wheelchair on the rear wheels for going down curbs or just for fun. The ease with which a system can be controlled (balanced) is related to the static stability of the system. The static stability is defined as:

$$\omega^2 = \frac{mg\ell}{J} \qquad [B]$$

where J is the mass moment of inertia at the center of gravity of the system about the direction perpendicular to the sideframe. For better wheelchair control during wheelchair balance the static stability should be reduced. Measurements of the value for the polar mass moment of inertia for a typical wheelchair + user of m = 90 kg was found to be J = 8.7 kg-m^2. In order to decrease the value of the static stability, Equation [B], one can increase J or decrease m and/or ℓ, where ℓ is the distance from the rear axle to the c.g. of the system. It is also shown that balancing a rod in the palm of the hand (inverted pendulum) is a mathematical problem similar to the wheelie balance problem, and a rod of length 1.56 meters is similar to a wheelchair + user system mass of 90 kg. However, balancing a rod is done primarily by using visual perception, whereas wheelie balance involves human joint proprioceptors and visual plus vestibular (inner ear) perception. Thus, a simple test of determining the shortest length of rod one can balance in the palm of the hand (plus measuring handrim force capability and simple reaction time) may indicate if a wheelchair user will find it easy to do a wheelie balance.

INTRODUCTION

Observation of the performance of a wheelchair athlete doing the wheelie maneuver for climbing a curb and balancing on the rear wheels suggests that both handrim force actuation and body manipulation are important. Once the athlete is balanced on the rear wheels, however, most of the reaction for

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maintaining balance seems to take place by handrim forces. Considering the latter situation as the simplest case of interest, it is possible to investigate mathematically the conditions for stability, using the techniques used to solve the inverted pendulum problem (5).

There have been a number of studies of biped stability which make use of the one degree and two degrees-of-freedom inverted pendulum problems (9,10). The general outcome is that a variety of feedback signals can stabilize biped models of this kind, and that there is a limit on the choice of signals necessary for stability. Thus, it seems reasonable to approach the wheelie stability problem by making use of the inverted pendulum analysis as developed in this paper.

MODEL

Figure 1 shows the geometry of the wheelchair wheelie balance problem. It will be assumed that the initial balance state has been obtained, but the results will also apply for "popping" a wheelie. Balancing a wheelchair on its rear wheels is a metastable equilibrium problem. When balance begins to be lost, the center of gravity of the wheelchair and user will rotate away from the equilibrium point, which is directly above the axle of the rear wheels. The user must react to the loss of balance by exerting a force on the handrims to move the rear wheel axle back under the center of gravity of the wheelchair + user. Point P must move with respect to the inertial coordinate system XY, shown to the left of the wheelchair.

Free body diagrams of the wheelchair + user and rear wheels with the appropriate forces and inertial reactions are shown in **Figure 2**. This is a two-degrees-of-freedom model in its simplest form; the differential equations of motion will be considered in the next section.

EQUATIONS OF MOTION

In the analysis of the problem it will be assumed that the pivot axis, which is the rear wheel axle, is essentially frictionless. At first, it will be assumed that the dynamic response of the person balancing the wheelchair is fast enough to allow us to neglect response delay. Later, response delay will be considered as a part of the man-machine control requirements. Also, it is assumed that wheel slip is zero, and that the force capability limit of the user is not encountered.

The analysis is similar to that presented by Cannon (3) or Elgerd (7) for the stick balance problem or inverted pendulum. At some instant of time, the

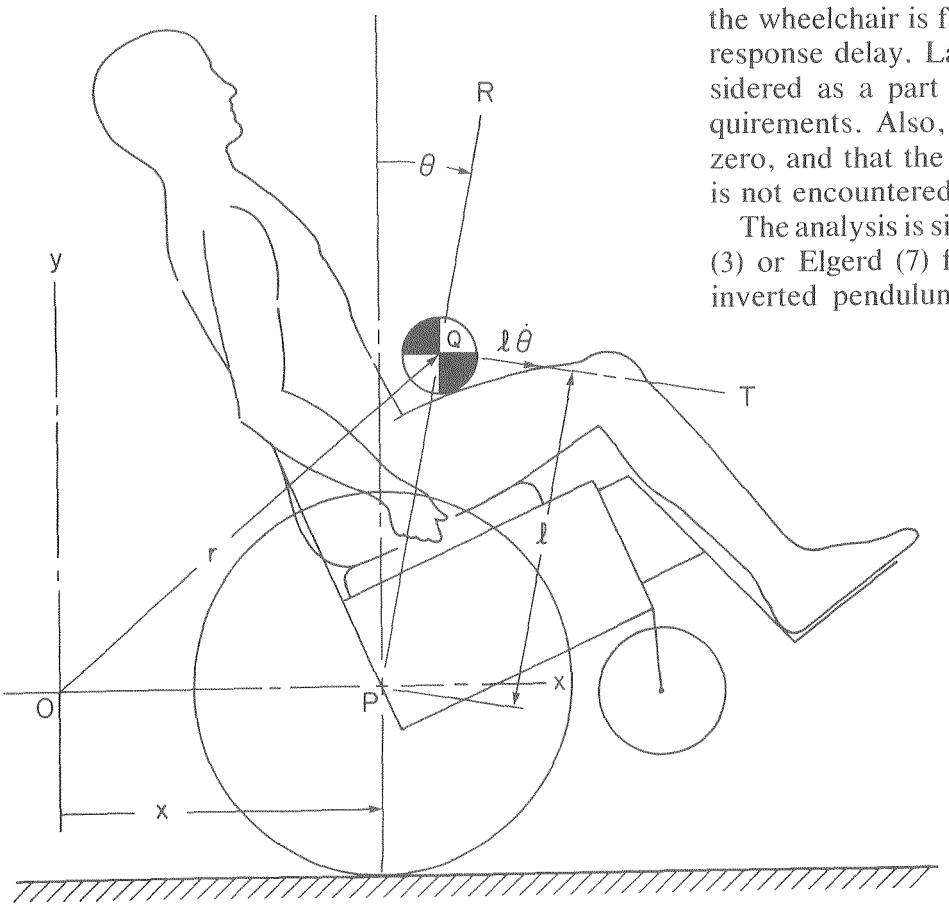

Figure 1
Geometry of wheelchair "wheelie" balance problem.

wheelchair + user has been disturbed from the equilibrium position by a small angle θ as shown in **Figure 1**. The acceleration of the wheelchair + user mass at point Q, the center of gravity, is complicated since the wheelchair + user mass rotates as well as translates. The vector position of point Q is:

$$r_Q = i_X X + (i_X \ell \sin\theta + i_Y \ell \cos\theta) \qquad [1]$$

where i_X and i_Y are unit vectors in the inertial reference frame.

Differentiating Equation [1] twice with respect to time gives the acceleration of point Q as:

$$a_Q = i_X(\ddot{X} + \ell\ddot{\theta}\cos\theta - \ell\dot{\theta}^2\sin\theta) + i_Y(-\ell\ddot{\theta}\sin\theta - \ell\dot{\theta}^2\cos\theta) \qquad [2]$$

Using the RT coordinate system in **Figure 2**, where i_T, i_R are unit vectors in the system's fixed reference frame, Equation [2] can be simplified to:

$$a_Q = i_X\ddot{X} + i_T\ell\ddot{\theta} + i_R(-\ell\dot{\theta}^2) \qquad [3]$$

Applying Newton's Second Law of Motion to the wheelchair + user free body diagram in **Figure 2**, with the aid of Equation [2], and taking the X component, gives:

$$m(\ddot{X} + \ell\ddot{\theta}\cos\theta - \ell\dot{\theta}^2\sin\theta) = f - f_h \qquad [4]$$

Again, consider the free body diagram of the wheelchair + user as shown in **Figure 2**. Taking moments about P and using Equation [3] gives:

$$J\ddot{\theta} + m\ell(\ddot{X}\cos\theta + \ell\ddot{\theta}) - mg\ell\sin\theta + R_h f_h = 0 \qquad [5]$$

where J is the mass polar moment of inertia of the wheelchair + user, less rear wheels, about the center of gravity (Q).

The problem can be simplified by using the small angle assumptions of $\cos\theta = 1$, $\sin\theta = \theta$, and $\theta\dot{\theta}^2 = 0$ in Equations [4] and [5]. Also, the substitution for $f - f_h = f_i = f_h R_h / R_w$ in Equation [4] is found by making use of the free body diagram analysis shown in **Figure 2**. It is also assumed that ε is negligible, and $f_i \cos\theta' = f_i$ since $\theta' < \theta$. The equation for the coefficient of rolling resistance is $\varepsilon = f_{RR} R_w / F_R'$, where for typical wheelchair tires $\varepsilon = 0.002$ m(0.078 in.), see reference (13). The wheel inertia force and moment are assumed negligible in comparison with the other forces and moments of the wheel.

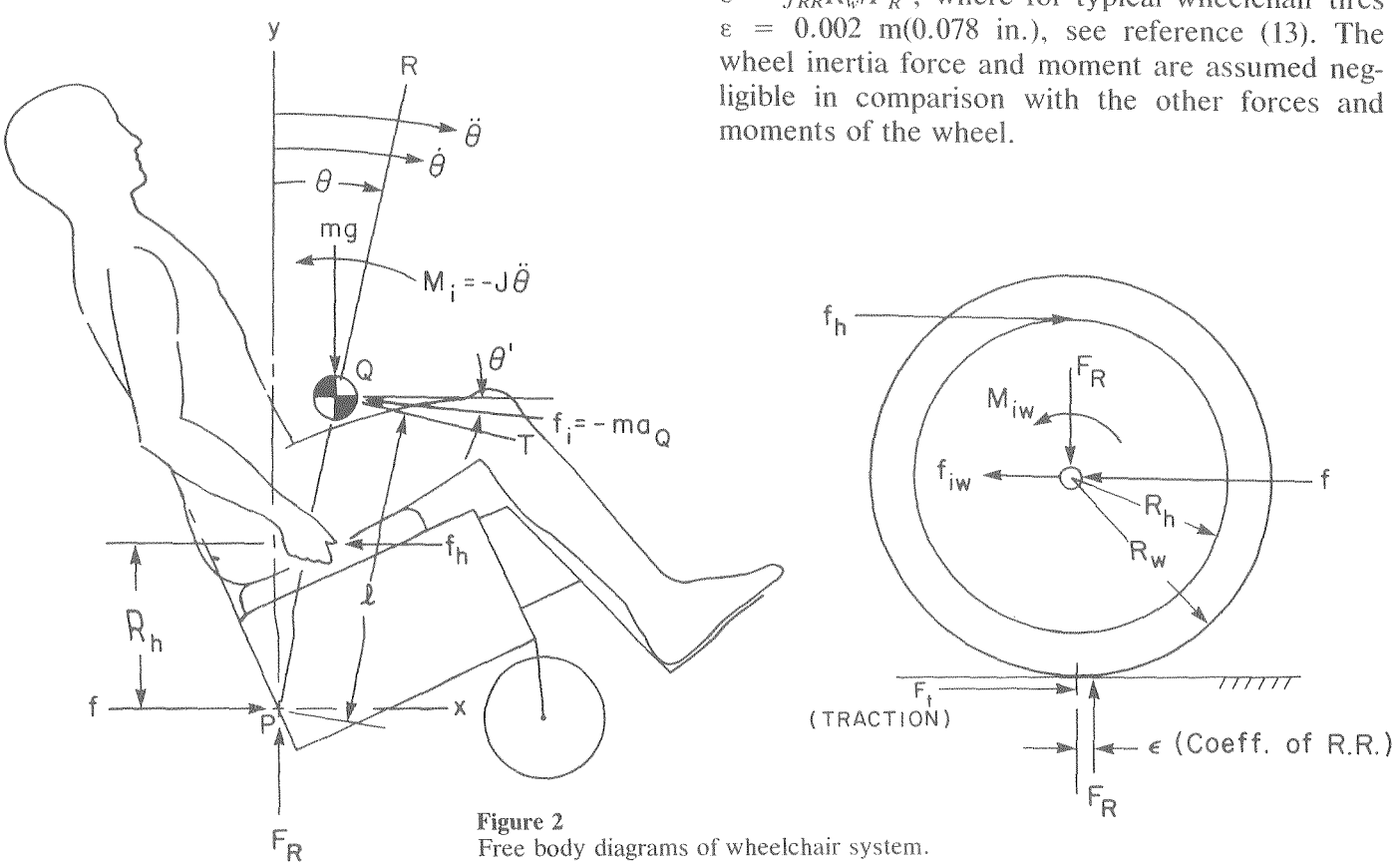

Figure 2
Free body diagrams of wheelchair system.

Performing the substitutions into Equations [4] and [5] gives the following pair of linear, second order, differential equations:

$$m(\ddot{X} + \ell\ddot{\theta}) = f_h \frac{R_h}{R_w} \qquad [6]$$

$$J\ddot{\theta} + m\ell(\ddot{X} + \ell\ddot{\theta}) - mg\ell\theta + R_h f_h = 0 \qquad [7]$$

Substituting $\ddot{X}$ from Equation [6] into Equation [7] gives:

$$J\ddot{\theta} - mg\ell\theta = -f_h\left(R_h + \ell\frac{R_h}{R_w}\right) \qquad [8]$$

Equation [8] is the fundamental equation of motion governing the wheelie problem for a wheelchair + user. This equation is of the same form as that found for the directional instability of a rear-caster wheelchair (12).

The right-hand term of Equation [8] is the torque, T, applied by the user, which acts about the center of gravity of the wheelchair + user system tending to restore the axle to the equilibrium position directly below the center of gravity. If the user does not apply a handrim force, Equation [8] and physical reasoning predicts that the wheelchair + user will angularly accelerate to the horizontal due to gravity, and balance is obviously not possible. The $-mg\ell\theta$ term is characteristic of an unstable system. If this term were positive, the equation would apply to an oscillatory system. For the purposes of discussion, it is convenient to substitute ω^2 for $mg\ell/J$ in Equation [8], although this term obviously does not represent the usual concept of natural frequency. Also, letting $T = f_h(R_h + \ell R_h/R_w)$ results in:

$$\ddot{\theta} - \omega^2\theta = -\frac{T}{J} \qquad [9]$$

The solution to Equation [9] is presented in Appendix A, and the application to the wheelie problem is given in the **RESULTS** section of this paper.

WHEELIE BALANCE TRAINING

Learning to do a wheelie balance was found to be easier on the new lightweight wheelchairs that have tipping bars to prevent falling backward. The authors found that removing the footrests at first made it easier to perform the wheelie maneuver. Adjusting the main axle location forward toward the plane of the center of gravity of the wheelchair + user also helped.

When one is initially trying to maintain a wheelie balance, there is a tendency to overreact. Young and Meiry (19) describe this aspect of manual control as "bang-bang" control. Bang-bang feedback control systems, also called off-on control, are used to control higher order systems such as the wheelchair. However, this method of control causes the wheelchair + user to rock back and forth continuously about the metastable equilibrium position of balance. If, instead, one makes an effort to apply a continuously variable control force to the wheelchair handrims, it is possible to control the wheelchair + user balance in a much smoother fashion, with more time spent nearer the wheelie balance point. It was found that several minutes of practice are required to develop a feel for exerting a continuously variable correction force, and that persons who have good motor coordination can learn to maintain a wheelie balance for several seconds after a practice period of 15 to 20 minutes.

We are interested in a more quantitative analysis of the wheelie balance problem, and a discussion of man-machine controls is considered next.

MAN-MACHINE CONTROL SYSTEMS

There are many common man-machine systems. For example, steering an automobile demonstrates the ability of man to act in a continuous adaptive control loop. Li (14) has presented a block diagram of the human sensors in vehicle control, also shown in **Figure 3**. The same control problems are encountered in using a wheelchair, and especially in being able to perform the wheelie maneuver.

Li (14) finds that in vehicle control, human visual motion rate perception is augmented by the vestibular sense of the ear which allows better performance than could be obtained using visual observations alone. Movements of the head cause the gelatinous mass in these vestibula to move, which deflects sensory hairs that stimulate associate nerve fibers. These nerves, in turn, inform the brain of the position of the head. Consequently, the brain sends control signals to the skeletal muscles to

control balance. In addition to vestibular perception, the eyes play a very important part in the control process. It is reported (11) that a person who has suffered damage to the vestibule of the ears can maintain normal balance as long as the eyes remain opened and body movement is performed slowly. Other sensory organs that aid in control are the proprioceptors associated in the joints of the body.

Consider **Figure 3**, where the eye observes a display and can sense position, velocity, and acceleration relative to stationary surroundings. At the same time the auditory vestibular system experiences the effects of gravity and acceleration and both systems aid in the control problem. There is a limit, however, to the complexity of the man-machine system which can be controlled. In the case of three-dimensional motion, bizarre effects of the auditory vestibular system known as disorientation can also occur. For the wheelchair wheelie problem, some indication of the difficulty of control is inferred by examining the inverted pendulum problem. Cannon (3, p. 707) finds that solid stability of an inverted pendulum requires a lead-network technique for control. The lead-network not only uses the control input of angular position, but, in addition, applies input from the angular velocity of the inverted pendulum. This suggests that a wheelchair user who cannot sense velocity would find doing a wheelie balance very difficult or impossible.

Additional information concerning the man-machine control problem is available from studies of pilot-aircraft performance. Young (18) has collected data on controllability of aircraft as a function of system parameters related to a second order differential equation, and his results are shown in **Figure 4.** The relevant differential equation is written as follows (15, p. 35):

$$\ddot{X} \pm 2\zeta\omega\dot{X} \pm \omega^2 X = \frac{F}{m} \qquad [10]$$

where $2\zeta\omega$ is the damping coefficient and ω^2 is the static stability. As mentioned with respect to Equation [9], when the sign preceding ω^2 is negative, the system is described as inherently unstable. It is possible to overcome the instability by proper input of the forcing function in Equation [10]. The wheelchair wheelie problem, with negative ω^2 and zero

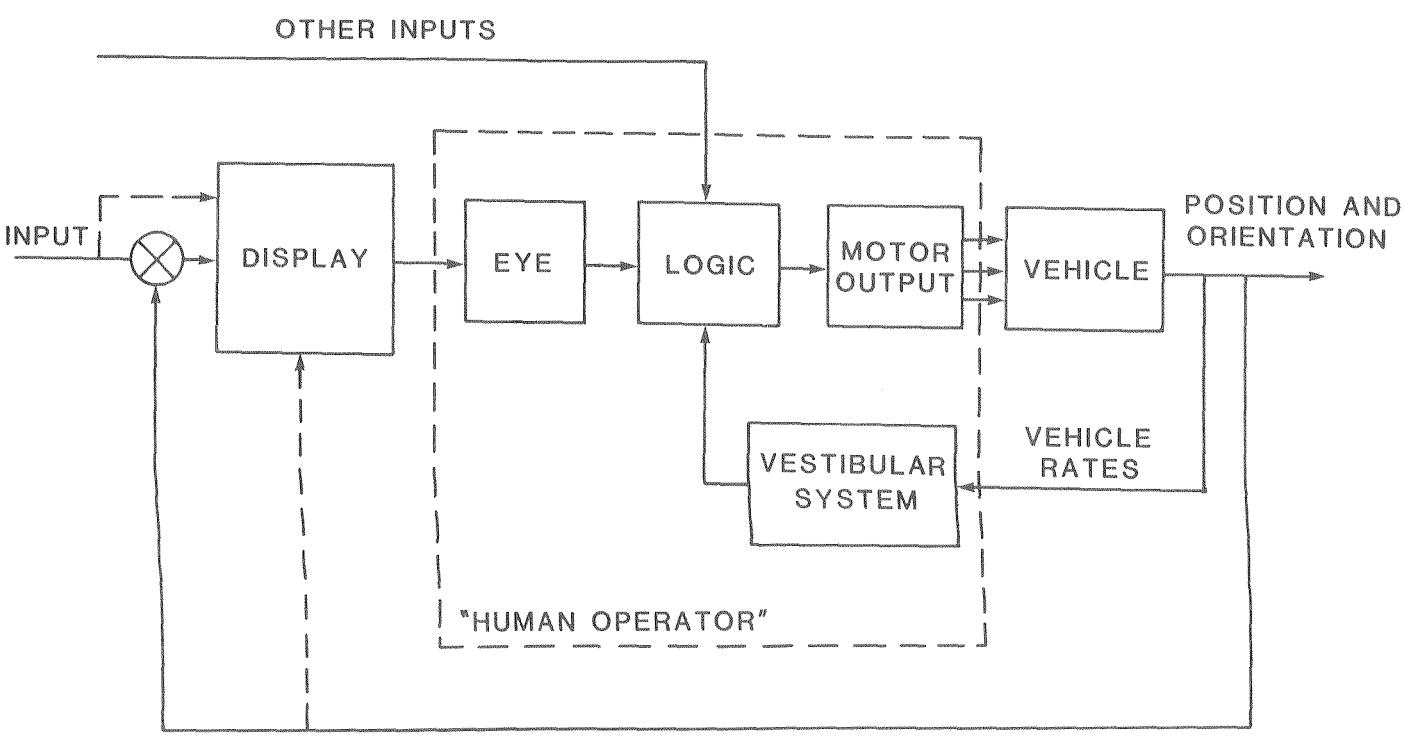

Figure 3
The Human Operator in Vehicle Control, after Li (8).

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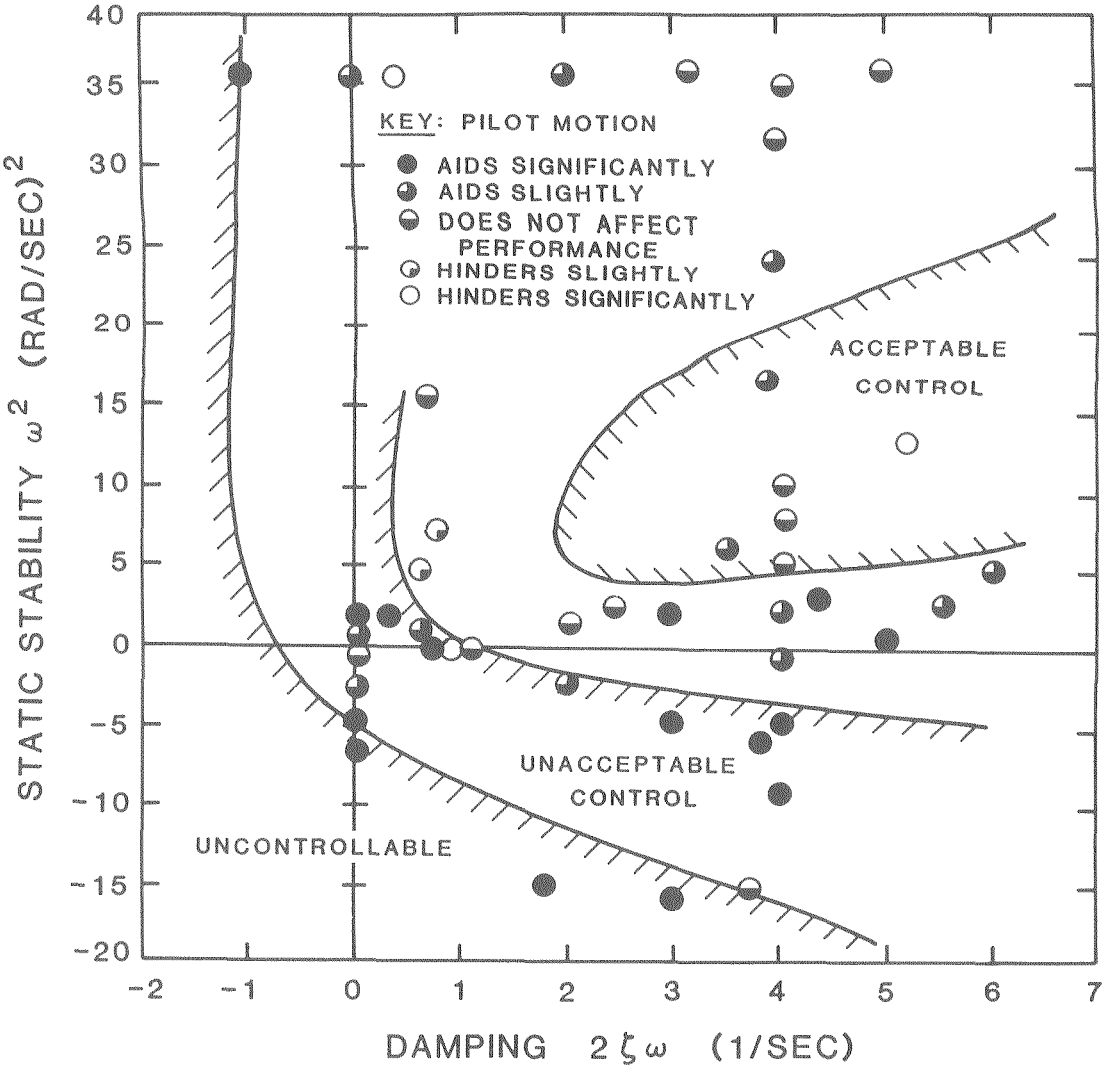

Figure 4
Evaluation of longitudinal handling qualities, high speed aircraft, launch vehicles, and VTOL aircraft, after Young (10).

$2\zeta\omega$ (see Equation [9]) falls in the region of great control difficulty. However, the wheelie problem involves fewer degrees-of-freedom than a pilot faces using aircraft controls. It is also shown in **Figure 4** that the pilot's own motion is important for control. Pilot motion refers to control when the pilot is in a simulator as opposed to controlling from outside the simulator.

Although a typical wheelchair wheelie problem falls in the unacceptable or uncontrollable region of **Figure 4**, it is apparent that the wheelchair wheelie balance maneuver is easily learned by many wheelchair users. This suggests that pilot-aircraft control information is not strictly applicable to the wheelchair wheelie problem, but it is believed that the trends are applicable.

For design purposes, **Figure 4** shows that changing the static stability toward the positive direction will improve control, and increasing the damping will also improve control.

RESULTS

Based on the solutions in **Appendix A,** calculations of wheelchair angular response versus time and handrim force are plotted. The problems considered are: 1) "popping" a wheelie, and 2) wheelie balance. Finally, a stick balance system with the same parameters as a 90 = kg wheelchair + user is presented as a simple device to determine if a user can develop sufficient wheelie balance skill.

1. **Popping a Wheelie**—When the wheelchair is in a horizontal position with all wheels on the ground and stationary (or moving), the user must exert a quick force on the handrims in the forward direction in order to cause the front of the wheelchair to lift (rotate about the rear axle). This effect is due to the inertia of the wheelchair + user. The maneuver is very effective for going over road curbs and other obstacles. The equation describing this motion is presented in **Appendix A,** as Equation [6A]. For the test wheelchair + user, the initial conditions and parameters needed to solve Equation [6A] are given in **Appendix B, Table B1.** The caster wheel force falls to zero rapidly and has a negligible effect on the solution. Substituting the appropriate values from **Table B1** into Equation [6A], the wheelie pop equation becomes

$$\theta = \frac{f_h}{528} - \frac{1}{2}\left(\frac{f_h}{528} - 0.5\right)(e^{6t} + e^{-6t}) \qquad [11]$$

A plot of Equation [11] is shown in **Figure 5,** where the angle Θ is shown to decrease with time when a sufficiently large handrim force is applied. For a particular user who can apply only a total handrim force (with two hands) of 264 N (59 pounds), the theory predicts that this force is too low to cause the front of the 90 kg (198 pounds) wheelchair + user to lift. Brubaker (2) has measured maximum handrim force for five subjects (four able-bodied and one paraplegic) who are young and athletic, and found the average handrim capacity ranged from 454 N (102 pounds) to 645 N (145 pounds) depending upon the handrim design. Thus, it would be predicted that these wheelchair users would find it easy to pop a wheelie from a stationary start. If the wheelchair user can also move his torso backward, thereby reducing Θ_0, so as to move the c.g. back toward the rear axle, the force necessary to pop a wheelie can be proportionately reduced.

2. **Wheelie Balance**—The wheelie balance problem, once the user has popped a wheelie and has brought the rear axle under the center of gravity of the wheelchair + user, will depend on the dynamic response capability of the user to maintain a balance position. Since the wheelie balance position of $\theta = 0$ is a point of metastable equilibrium, any slight deviation will cause the wheelchair to fall. The user must respond properly and in a timely manner with sufficient handrim force depending on the direction

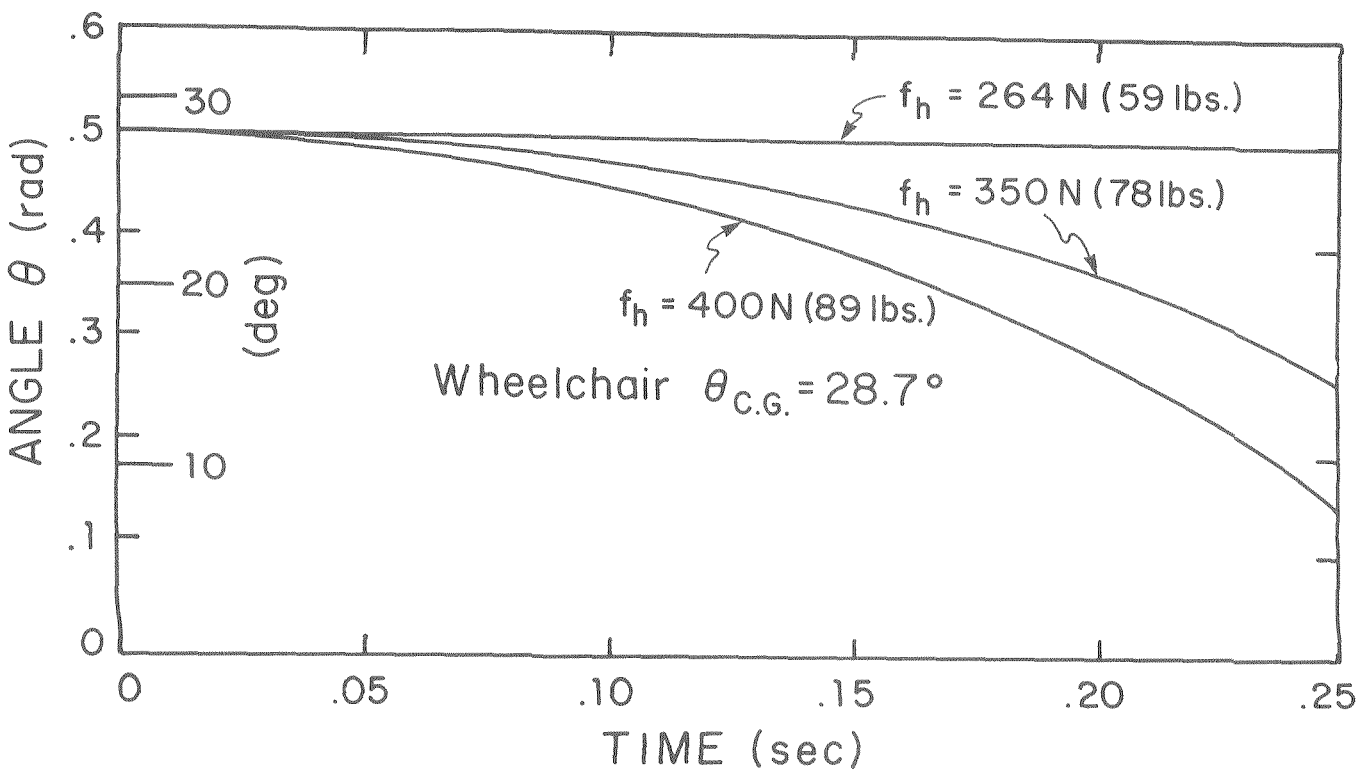

Figure 5
Popping a Wheelie: Angle θ vs. time for handrim force applied.

of the fall. As the wheelchair + user is restored to the equilibrium wheelie balance position, the user must be able to adjust the handrim force so as not to overshoot the balance position.

The balance problem is analyzed in two parts. First, the falling motion of the wheechair + user is considered during the delay reaction time of the user, and second, the handrim force necessary to recover a balanced position is considered.

Factors affecting reaction time of human subjects are discussed by Frost (8), who finds that reaction time to a discrete stimulus is one of the least understood facets of human behavior. Every variable affecting human behavior, such as fatigue, motivation, etc., will affect reaction time. Frost gives a range of reaction times for various senses, where those that are applicable to wheelie balance are touch (with t_0 = 0.11 to 0.15 seconds) and vision (with t_0 = 0.15 to 0.20 seconds). A study by Do, Bouisset, and Moynot (4) of 8 able-bodied and 12 paraplegic (T4 level) subjects performing a simple arm movement task of moving a weight on a table, reported that the average reaction delay time measured from first muscle twitch to beginning to move the weight, for able-bodied subjects, was 0.078 seconds, while for the group of paraplegics it was 0.177 seconds. A simple reaction-time experiment of lifting a weight at a light signal gave results in the 0.20 to 0.25 seconds range for several able-bodied students in this laboratory. The data suggest that a wheelchair user will have a reaction time of no less than 0.1 seconds and, probably, as long as 0.2 to 0.3 seconds.

For the calculation, it is assumed that an arbitrary but very small deviation from the balance position occurs, θ_0 = 0.1 rad (5.7 degrees) with $\dot{\theta}$ = 0 and T = 0. The equation describing the angular rotation of the wheelchair + user is given in **Appendix A** as Equation [6A]. Substituting the system parameter values given in **Table B1** into Equation [6A] gives

$$\theta = 0.05(e^{6t} + e^{-6t}) \qquad [12]$$

where $t < = t_0$. After a time corresponding to the reaction time of the user, the user is assumed to apply a handrim force to counteract the fall from wheelie balance. The motion of the wheelchair + user is now governed by the solution given in **Appendix A,** Equation [9A]. For this part of the analysis, it is mathematically convenient to let

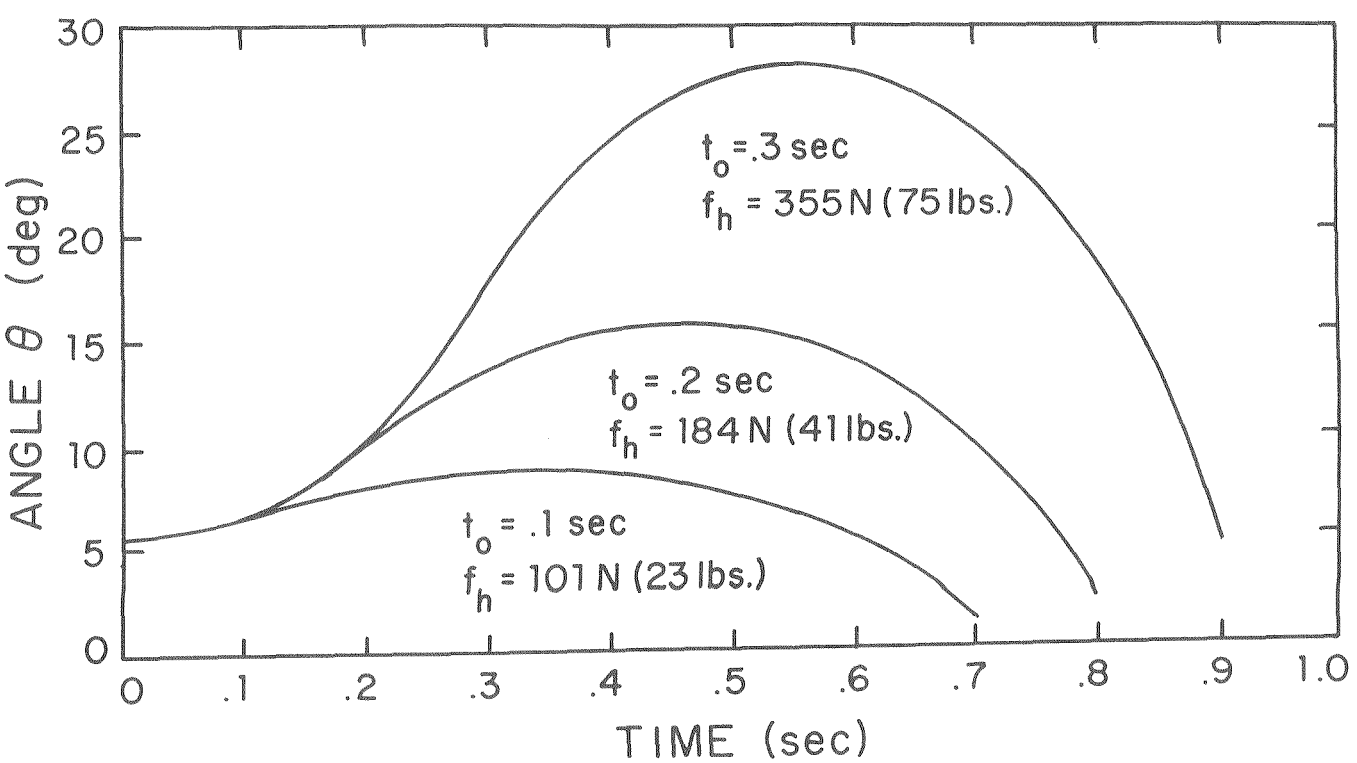

Figure 6
Wheelie balance recovery angle vs. time for three values of human response delay time.

$$\frac{T}{J\omega^2} = n\theta_0 e^{\omega t_0} \qquad [13]$$

where $1 < n < 2$. Substituting Equation [13] into Equation [9A] results in

$$\theta = 0.05e^{6t_0}[(1-n)e^{6t'} - (n - e^{-12t_0})e^{-6t'} + 2n] \qquad [14]$$

where $t' = 0$ at t_0, i.e., $t = t_0 + t'$.

Since $T/(J\omega^2) = f_h/528$ rad for the test wheelchair + user system, Equation [13] can also be used to calculate the handrim force. By trial and error, letting $n = 1.05$ gives reasonabale results, and Equations [12] and [13] can be plotted for three values of user reaction time as shown in **Figure 6.** In **Figure 6,** time proceeds from the beginning of the fall away from equilibrium balance and there is no user handrim response until the reaction time is reached. Beyond the reaction time, it is assumed that a constant handrim force is applied to restore the wheelchair + user to the balance position. The user must be able to adjust the restoring handrim force so as not to overshoot the balance point. However, this aspect of the problem has been omitted as being beyond the scope of the paper.

Figure 7 shows the force required to produce the response shown in **Figure 6** as a function of reaction time. As would be obvious from consideration of the model, the shorter the delay reaction time the smaller the handrim force necessary to recover wheelie balance.

3. **Inverted Pendulum**—The fact that the wheelie balance problem has the same governing differential equation as the inverted pendulum suggests a similarity between hand-balancing a vertical rod and wheelchair + user wheelie balance. The governing parameter in Equation [9] is $\omega^2 = mg\ell/J$. For a rod of length $L = 2\ell$, the polar mass moment of inertia about its c.g. is given by

$$J_{\text{rod}} = m\frac{\ell^2}{3} \qquad [15]$$

Solving for rod length, L, and using $\omega^2 = 36$, one obtains

$$L = 6\frac{g}{\omega^2} = 1.56 \text{ meters} \qquad [16]$$

as the length for a rod with similar stability characteristics.

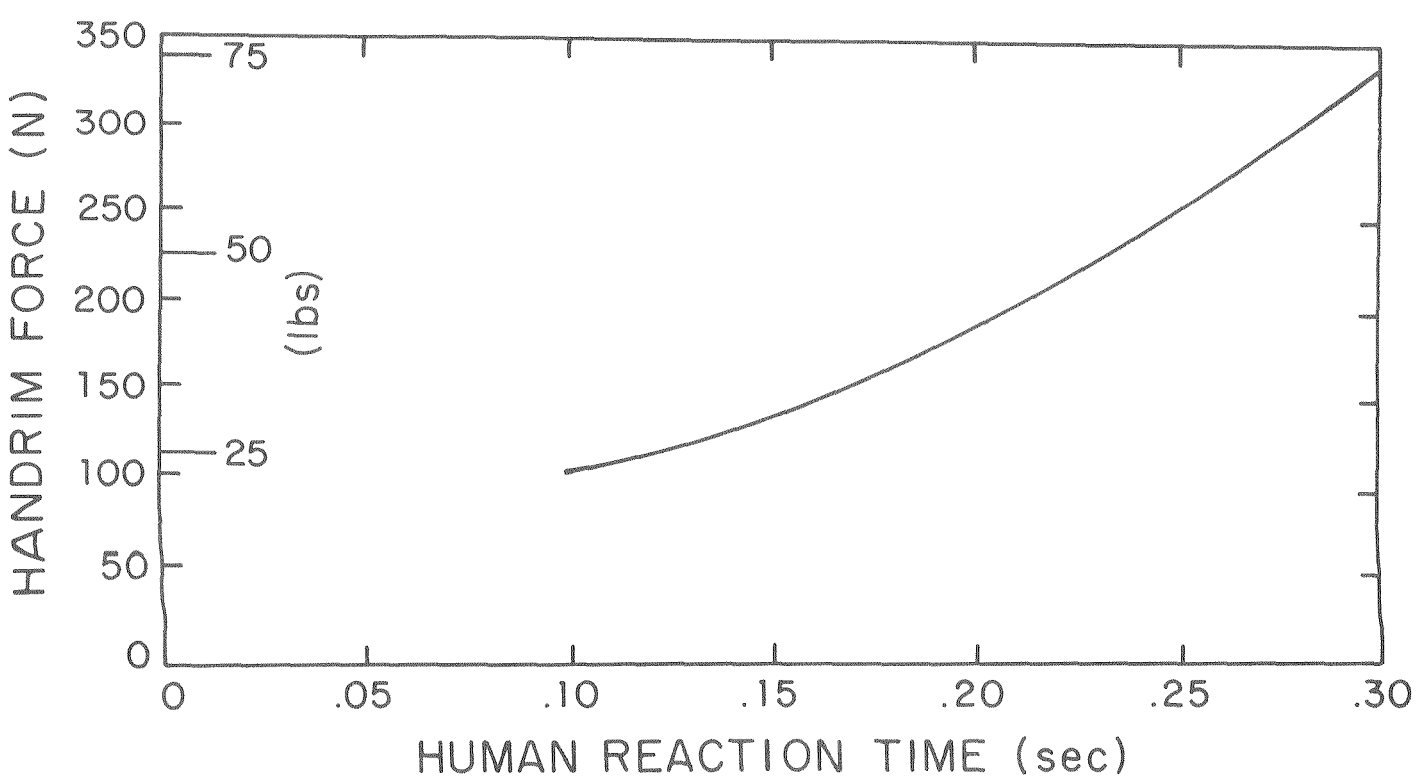

Figure 7
Wheelie balance recovery handrim force vs. reaction time.

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Figure 8 shows a schematic of this problem. Limited testing with several able-bodied graduate students showed that a 1.56 m (61 inch) rod is fairly easy to hand-balance while standing or sitting, and that some sudents found it possible to balance a 1/3 m (13 inch) rod after a number of practice sessions. These students were also able to learn to do a wheelchair wheelie balance, more or less, after several practice sessions. Later, with much experience, two students found thay could do a wheelie balance with their eyes closed . An active paraplegic was also able to do a wheelie balance with eyes closed .

The shape of the rod has not been studied extensively, but it was found easier to balance a thin blade (8 cm wide) than a thin rod, both of the same length. The blade tends to limit the instaility to one dimension, which is similar to the wheelie balance problem.

Although auditory vestibular senses are not involved, the visual perception needed to hand-balance a rod, along with a measurement of one's handrim force capacity and reaction time, may be measures of a user's potential ability to do a wheelie balance. More work is needed to verify these suggestions concerning the applicability of similar system measurements.

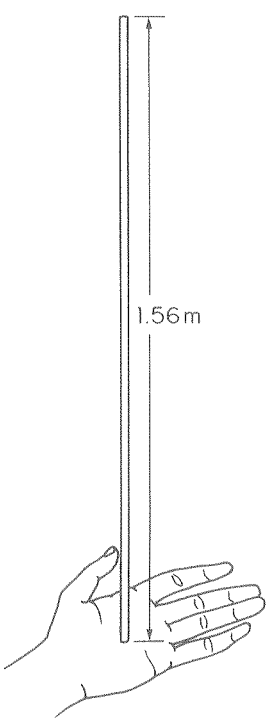

Figure 8
Similar wheelie balance system.

WHEELCHAIR DESIGN EQUATIONS

It is always useful to have simplified equations that will give the general trends or "rules of thumb" for design purposes. One aspect of interest in this paper is the effect on design of the force necessary to pop a wheelie. This information is contained in Equation [6A] along with the measurements of the system given in **Figure 9.**

It is assumed that it is desired to pop a wheelie in order to place the caster wheels on a 10-cm curb. This involves applying a handrim force so as to raise the front of the wheelchair sufficiently. Starting with Equation [6A], and considering the problem of curb climbing, the initial conditions are the "c.g. angle" of $\theta_{c.g.} = 0.5$ rad (28.7 degrees) and $\dot{\theta}_p = 0$. The final condition is the "pop-angle," $\theta_p = 0.20$ rad (17.2 degrees) for a 10-cm curb (4 inches). By adding and subtracting $\theta_{c.g.}$ on the right-hand side of Equation [6A] the following useful equation is obtained.

$$\theta_{c.g.} - \theta_p = \left(\frac{T}{J\omega^2} - \theta_{c.g.}\right)\left[\frac{1}{2}\left(e^{\omega t} + e^{-\omega t}\right) - 1\right] \qquad [17]$$

Examining Equation [17], we see that $(\theta_{c.g.} - \theta_p)$ is positive, and the only way the right-hand side will be positive is when

$$\frac{T}{J\omega^2} \geqq \theta_{c.g.} \qquad [18]$$

An additional condition is needed and it can be obtained from **Figure 5.** There, it is shown that the handrim force required to pop a wheelie in a reasonable length of time (0.25 seconds) is about 133 percent of the minimum force. Substituting for $T/(J\omega^2)$ in Equation [18] from Equations [8] and [9], and using dimensions for the test wheelchair, gives

$$f_h = 0.8 \text{ mg } \theta_{c.g.} \qquad [19]$$

Equation [19] shows that reducing the mass and/or the "c.g. angle" of the wheelchair will reduce f_h. It can be seen from **Figure 9** that $\theta_{c.g.}$ can be reduced by moving the rear axle position forward toward the plane of the center of gravity.

Another design aspect of this paper is wheelie

balance. As shown in **Figure 4,** the ease with which a system can be controlled is related to the static stability. The wheelchair + user has a negative static stability given by

$$\omega^2 = \frac{mg\ell}{J} = \frac{g\ell}{j} \qquad [20]$$

where mj = J. Reducing ω^2 by decreasing ℓ, the distance from the axle to the c.g., will lead to easier control of wheelie balance. This result for the wheelchair system appears to be contrary to practical experience, since a longer rod is easier to balance. But the length of a rod affects the specific polar moment of inertia of the rod, such that a longer rod gives a smaller ω^2. One should consider the wheelchair problem as being the balancing of a fixed mass on the end of an essentially massless rod. In that case, shortening the length of the massless rod will improve the balance control, in accordance with Equation [20].

CONCLUSIONS

A mathematical analysis of wheelchair wheelie performance based on the classical inverted pendulum problem is presented. The inverted pendulum appears to represent a reasonable model for wheelchair wheelie maneuvers. The results can be used to predict handrim force necessary to pop a wheelie as well as maintaining wheelie balance.

Balancing an inverted pendulum rod of 1.56 m length in the palm of the hand is shown to be a problem mathematically similar to the wheelie balance problem for a 90-kg wheelchair + user. This, plus the wheelie balance solution, suggests that simple tests of determining the length of the shortest rod one can hand-balance, measurement of handrim force capability, and measurement of human reaction time, may indicate whether a person will be able to perform the wheelie balance maneuver. However, more work is needed to establish such a testing protocol.

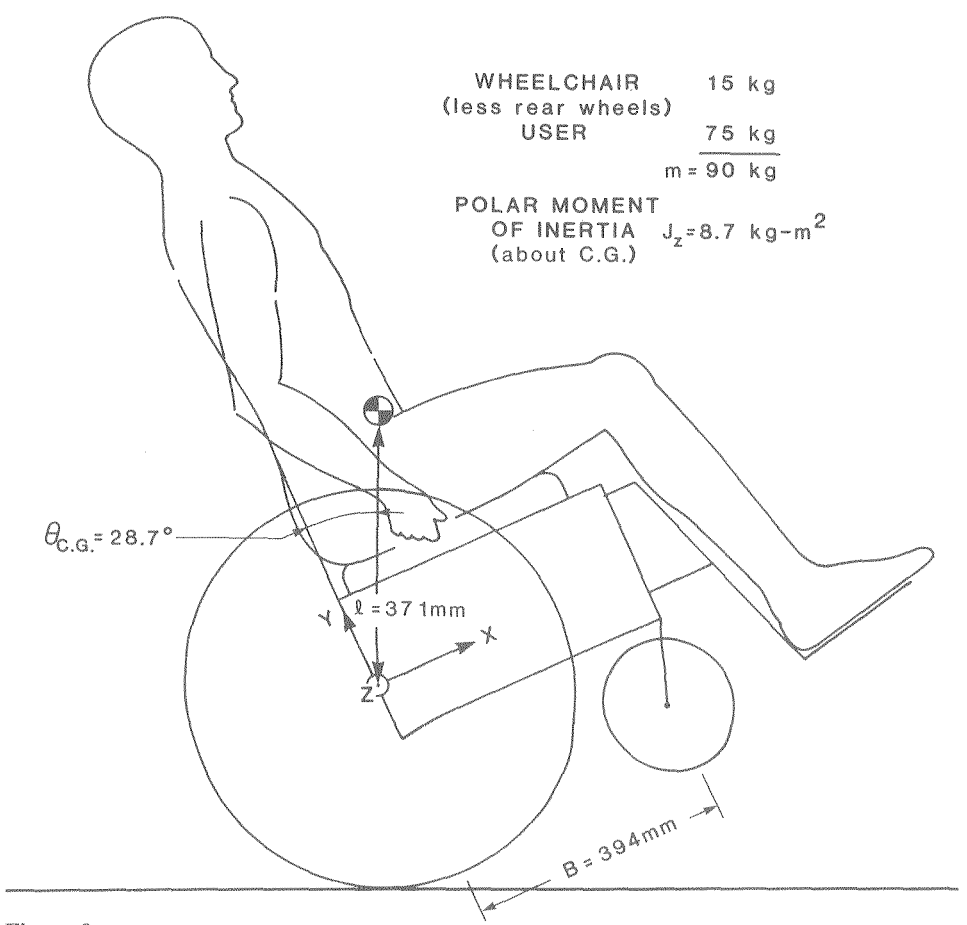

Figure 9
Typical wheelchair + user wheelie parameters.

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APPENDIX A: DIFFERENTIAL EQUATION SOLUTIONS

Consider the solution to Equation [9], where Equation [9] represents the rotation of the wheelchair + user on the rear axle during a wheelie maneuver. Equation [9] is repeated as [1A].

$$\ddot{\theta} - \omega^2\theta = -\frac{T}{J} \qquad [1A]$$

Where

$$\omega^2 = mg\frac{\ell}{J} \qquad [2A]$$

and

$$T = f_h\left(R_h + \ell\frac{R_h}{R_w}\right) \qquad [3A]$$

Assuming the handrim force f_h is a constant, the simplest case of interest, Equation [1A], has the well-known solution (see reference 1, pages 2–50), of

$$\theta = c_1 e^{\omega t} + c_2 e^{-\omega t} + \frac{T}{J\omega^2} \qquad [4A]$$

The boundary conditions at $t = 0$ are

$$\theta = \theta_0 \quad \text{and} \quad \dot{\theta} = \dot{\theta}_0 \qquad [5A]$$

By solving Equation [4A] with the initial conditions listed in Equation [5A], the solution equation describing the angular response of the wheelchair + user is

$$\theta = \frac{1}{2}\left(\theta_0 + \frac{\dot{\theta}_0}{\omega} - \frac{T}{J\omega^2}\right)(e^{\omega t} + e^{-\omega t}) - \frac{\dot{\theta}_0}{\omega}e^{-\omega t} + \frac{T}{J\omega^2} \qquad [6A]$$

When the wheelchair is horizontal, and when it is either stationary or moving, Equation [6A] can be used to predict the handrim force necessary to pop a wheelie.

The next problem of interest occurs when the wheelchair is at the wheelie balance point and is disturbed from equilibrium. In this case Equation [6A] describes the angular motion of the wheelchair + user up to the time when a corrective handrim force is applied.

As wheelie balance is lost, the user will apply a corrective handrim force after some reaction delay time. A solution to equation [4A] for the problem after a handrim force is applied is found by using the following initial conditions. Substituting the reaction time t_0 into Equation [6A] and letting $\dot{\theta}_0 = 0$ and $T = 0$, gives

$$\theta = \frac{1}{2}\theta_0(e^{\omega t_0} + e^{-\omega t_0}) \qquad [7A]$$

Taking the derivative of Equation (6A) and letting $\dot{\theta}_0 = 0$ and $T = 0$ results in

$$\theta = \frac{1}{2}\theta_0\omega(e^{\omega t_0} - e^{-\omega t_0}) \qquad [8A]$$

Using Equations [7A] and [8A] as initial conditions in Equation [4A] leads to the following solution

$$\theta = \frac{1}{2}\left(\theta_0 e^{\omega t_0} - \frac{T}{J\omega^2}\right)e^{\omega t'} + \frac{1}{2}\left(\theta_0 e^{-\omega t_0} - \frac{T}{J\omega^2}\right)e^{-\omega t'} + \frac{T}{J\omega^2} \qquad [9A]$$

where t_0 = user reaction time, and $t' = 0$ at $t = t_0$, i.e., $t = t_0 + t'$.

There are other, more complicated, cases of interest which would involve the man-machine as a closed-loop feedback control system. This would require consideration of the handrim force as a function of position and velocity. For our purposes, it is sufficient to consider the simpler problems in which the handrim force is a constant, at least at the beginning of any response.

APPENDIX B: WHEELCHAIR-USER PARAMETERS

The analysis of wheelchair wheelies performance requires several wheelchair + user measurements not normally available from the manufacturers' literature. These parameters are the location of the center of gravity, and the mass moment of inertia about the z-axis.

The center of gravity and polar moment of inertia of the wheelchair with the large wheels removed was determined using the torsional pendulum method. For this, the wheelchair was suspended on its side at the c.g. by a steel rod 3.18 mm O.D., length of 851 mm, and a measured torsional spring constant of 0.889 N–m/rad. The mass and natural frequency were measured to predict the inertia of the system.

A 75 kg ISO Dummy (17) in a vertical-back seated position (**Figure 9**) was measured for user parameters. The method used was to calculate the c.g. and moments of inertia of the ISO Dummy components and calculate a composite c.g. and moment of inertia. These calculations were prepared by Duffey (6).

Using the parallel axis theorem, the c.g. and inertia of the system were calculated. The results of the calculations are shown in **Figure 9** and listed in **Table B1.**

Table B1:

Test Wheelchair Parameters (see Figure 9)

m	90 kg (198 pounds)
ℓ	371 mm (14.6 inches)
R_w	305 mm (12 inches)
R_h	279 mm (11 inches)
J_z	8.7 kg–m^2 (6.4 lb-ft-s^2)
$\theta_{c.g}$	0.5 rad (28.7 degrees)
ω^2	37.6 s^{-2}
B	394 mm (15.5 inches)

NOMENCLATURE

a_Q	acceleration at point Q, m/s^2
B	wheelchair wheelbase, m
F_R	rear wheel force, N
F_t	tractive force, N
f_{RR}	rolling resistance force, N
f_i	inertial force, N
f_{iw}	wheel inertial force, N
f	rear axle force, N
f_h	handrim force, N
g	acceleration of gravity, 9.807 m/s^2
i_R, i_T, i_X, i_Y	unit vectors, m
J	polar moment of inertia, kg–m^2
j	specific polar moment of inertia, kg–m^2/kg
L	length of stick, m
ℓ	axle to c.g. length, m
M_i	inertial moment, newton–meters (N–m)
M_{iw}	wheel inertial moment, N–m
m	wheelchair + user less rear wheels, mass, kg
P	rear axle point
Q	system center of gravity point
R	coordinate, m
R_h	Handrim radius, m
R_w	rear wheel radius, m
r_Q	instant radius to Q, m
T	coordinate, m; or torque, N–m
t	time, (seconds) s
t′	time after t_0, s
t_0	human reaction time, s
X	coordinate, m
$\dot{X}$	velocity, m/s
$\ddot{X}$	acceleration, m/s^2
Y	coordinate, m
Z	coordinate, m
ε	coefficient of rolling resistance, m
ζ	damping factor
$\theta_{c.g.}$	c.g.-angle, rad (see Equation 6)
θ_p	pop-angle, rad (see Equation 6)
θ'	inertia force angle, rad (see **Figure 2**)
θ	angular position, rad
$\dot{\theta}$	angular velocity, rad/s
$\ddot{\theta}$	angular acceleration, rad/s^2
ω	angular frequency, rad/s

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