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Application of adaptive digital signal processing to speech enhancement for the hearing impaired

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Abstract—A major complaint of individuals with normal hearing and hearing impairments is a reduced ability to understand speech in a noisy environment. This paper describes the concept of adaptive noise cancelling for removing noise from corrupted speech signals. Application of adaptive digital signal processing has long been known and is described from a historical as well as technical perspective. The Widrow-Hoff LMS (least mean square) algorithm developed in 1959 forms the introduction to modern adaptive signal processing. This method uses a "primary" input which consists of the desired speech signal corrupted with noise and a second "reference" signal which is used to estimate the primary noise signal. By subtracting the adaptively filtered estimate of the noise, the desired speech signal is obtained. Recent developments in the field as they relate to noise cancellation are described. These developments include more computationally efficient algorithms as well as algorithms that exhibit improved learning performance.

A second method for removing noise from speech, for use when no independent reference for the noise exists, is referred to as single channel noise suppression. Both adaptive and spectral subtraction techniques have been applied to this problem—often with the result of decreased speech intelligibility. Current techniques applied to this problem are described, including signal processing techniques that offer promise in the noise suppression application.

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INTRODUCTION

Hearing impairment is not only the most prevalent communicative disorder, it is also the number one chronic disability affecting people in the United States. A major complaint of those with hearing impairments is a reduced ability to understand speech in everyday communication in a noisy environment. Even with the absence of hearing impairment, the addition of background noise can signifiantly reduce the intelligibility of speech. In 1956, Widrow proposed an adaptive filter as shown in **Figure 1** which can be used to reduce interference when a second sample of the noise is available. This technique was developed at Stanford University in 1959 and applied to a pattern-recognition scheme known as Adaline. In 1965 the first adaptive noise cancelling system was built by two students at Stanford University. In 1972, the first all-digital adaptive filter was built by McCool and Widrow at the Naval Undersea Center in Pasadena, California. In 1975, several applications of the LMS algorithm were presented which included adaptive noise cancelling and noise suppression (32). The LMS algorithm was simple—both in the number of calculations required for it's update and in it's derivation—and robust in a number of applications. An adaptive feedback constant, μ,

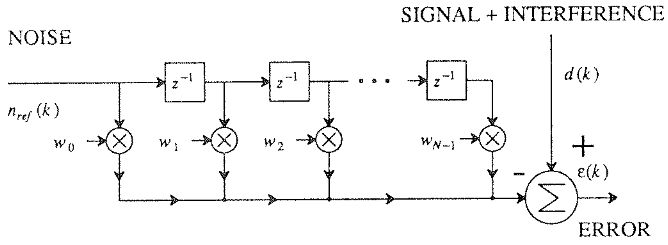

Figure 2.
An adaptive LMS filter.

tion from the desired optimal filter results. By reducing the size of the feedback coefficient μ in Equation [3], the misadjustment can be made arbitrarily small. However, since the adaptation time is inversely proportional to μ, a large μ for fast learning is required in applications where the statistics of the input signals vary with time. This selection, however, results in increased misadjustment or residual error. One may optimize the choice of μ in cases of these nonstationary inputs by selecting a feedback constant such that the error due to tracking the nonstationarity in the signal just equals the misadjustment or error residual that occurs because of the constant updates that occur after the filter coefficients, $w_p(k)$, have converged to their desired value (31).

Since the filter parameters adapt in such a way as to provide an estimate of $n_{pri}(k)$ from the reference signal $n_{ref}(k)$, the task in applying the algorithm to noise cancellation becomes one of providing sufficient degrees of freedom that an acceptable solution may be obtained. The following design or selection criteria must be followed:

1. The number of digital filter stages (N in Equation [1] should be selected so that N times T, the sample period of the digital system, is larger than the impulse response or reverbertion time of the acoustic environment. For small rooms, typical filter lengths are on the order of 1,000 to 2,000 stages for a sample rate of 10 kHz. It is not uncommon to require 8,000 to 16,000 stages of adaptive filtering for moderate size rooms at a sample rate of 10 kHz.
2. The selection of the feedback constant μ is made according to the desired adaptation rate. The choice of μ for a given adaption rate with common broadband noise interference is given as

$$\mu = \frac{1}{4\tau\sigma_n^2} \qquad [4]$$

where σ_n^2 represents the variance of the reference noise process, $n_{ref}(k)$, and τ is the desired adaptation time in samples. In no case should the upper limit for μ exceed $\sigma_n^2/2$ or instability will result.
3. A time delay must be inserted into either the primary or reference channel as necessary to insure that the desired filter is causal. That is, the reference noise signal should enter the reference input to the adaptive filter during the same processing period in which the correlated primary interference arrives at the summing junction in **Figure 1**.
4. In cases where there is leakage, care must be taken to minimize leakage of the desired speech signal into the reference input. In such cases, the interference cancellation is limited to the ratio of the interference signal to speech signal in the reference.

Following these four procedures in the use of an adaptive filter has resulted in interference reductions corresponding to speech enhancement of up to 60 dB., but more typically 30 dB.

A variant of the LMS algorithm that provides the ability to adapt the feedback constant μ was developed by Harris (17). In this approach known as the VS adaptive algorithm, a separate $\mu_p(k)$ is calculated for each stage of the filter. Results of the VS algorithm applied to noise cancellation show a speedup in adaptation time of up to a factor of 50 without increasing the residual error while maintaining both speech quality and intelligibility.

Frequency Domain Adaptive Filters

One of the drawbacks of the LMS adaptive filter in processing speech signals is the error criterion, which is selected for the minimization of mean square error and which results in an adaptation process that treats frequency regions of higher energy content before adapting to regions of low energy. As a result, the lower frequency regions of the speech spectrum receive an inordinate amount of attention at the cost of the high-frequency speech regions—which contain much of the intelligibility. As a second consideration, users of adaptive filters are always anxious to find more efficient computational techniques to perform the adaptive filtering tasks. As a result, several papers describing frequency-domain implementations for adaptive filtering have been presented (7,9,10,12,26,30). One of corrupted by interference when no independent or second reference is available is referred to as noise suppression.

In 1978 Sambur (27) proposed to apply an LMS version of the digital adaptive filter described in the foregoing but using reference delays equal to one or two voice pitch periods. Sambur reasoned that in the speech component of the corrupted signal there would be strong correlation between the primary and delayed reference inputs. He reported improved SNR and speech quality but did not claim improved intelligibility. In fact, the time domain LMS adaptive filter concentrates its computational power first on those frequencies in the signal with the highest energy to minimize mean square error, (i. e., pitch and pitch harmonies where little intelligibility is carried) and finally on frequencies with the least energy (i.e., high frequency sounds where most of the information in speech is carried). This results in an output that sounds like muffled speech deplete of high-frequency information.

The muffling effect appears to be present in most speech enhancement systems, prompting the statements by Lim (21) and Schafer (29) that the various speech enhancement systems appear to improve the subjective speech quality but not speech intelligibility, and that successful approaches must exploit more knowledge about the information-bearing elements of speech.

Time Domain Filters for Noise Suppression

Figure 3 describes the application of the LMS Widrow-Hoff time domain adaptive algorithm to the problem of noise suppression. This filter is attractive because of its relative simplicity. Sambur (27) proposed the time domain implementation of the adaptive filter shown in **Figure 3**, which employs the LMS Widrow-Hoff algorithm in Eqs. 1–3 for the filter coefficient updates. Implementation of this LMS adaptive filter results in three deficiencies:

1. The speech spectrum is distorted, with the low-frequency region enhanced due to the high energy content at the low frequencies. This is a result of the mean square error criterion.
2. Unvoiced speech sounds are eliminated in the signal processing by large delays which are typically a pitch period. This results in confusion between such words as net, nets, next, etc. leading to reduced intelligibility.

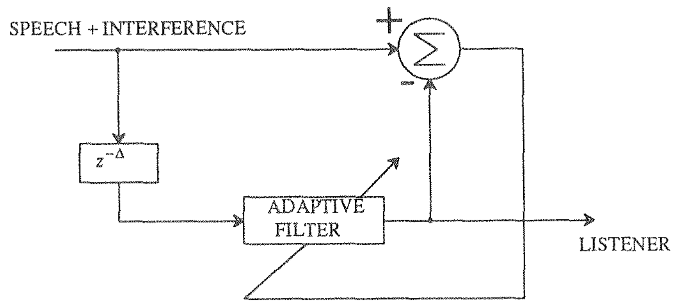

Figure 3.
Time domain adaptive LMS noise suppressor.

3. Reverberation is introduced because the LMS algorithm in Eqs. 1–3 responds to minimize mean square error and will leave large values for $w_p(k)$ when $n_{ref}(k\text{-}p)$ becomes small or is zero—as is the case during the silent portions of speech. The sound introduced is reminiscent of listening to a sea-shell and hearing that reverberant background.

The general behavior of the adaptive noise suppressor with the inherent problems just described may be seen in **Figure 4** which shows the spectrum of noise-free speech before and after processing with the LMS algorithm as proposed by Sambur. Here it is clear that the high-frequency information above about 1.2 kHz, which is evident in **Figure 4a**, is gone in **Figure 4b**.

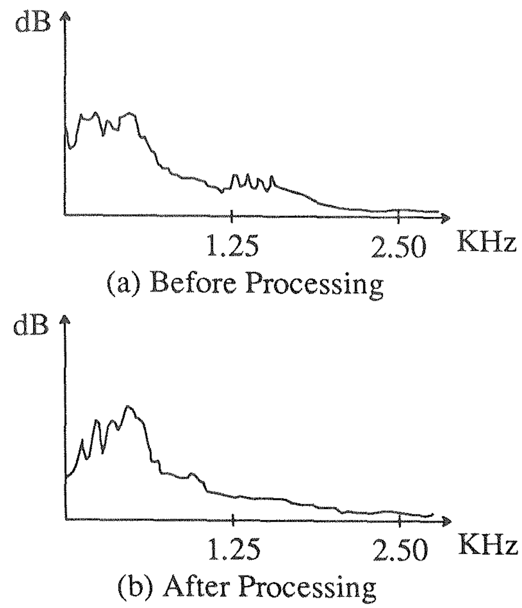

Figure 4.
Noiseless speech before and after processing by an adaptive LMS filter.

There are several approaches to reducing the deleterious effects introduced by time domain adaptive processing. By forcing increased processor attention to high frequencies, the spectral distortion may be made acceptable. A second problem is observed during the change from speech to silence between words. The time domain adaptive processor stops updating the filter weights when the input signal power decreases significantly (during speech silence) so the filter "remembers" the weights from the prior speech. As the next word begins, an annoying echo or synthetic reverberation effect is produced.

One solution for overcoming these deleterious effects has been presented by Andersen (6). His treatment included the following:

1. **Pre-whitening.** The general behavior of the adaptive LMS filter in processing both high frequency-low energy and low frequency-high energy portions of the speech spectrum is illustrated in **Figure 4.** In **Figure 4b** it is clear that the high frequency information bearing elements of the speech sample have been removed. By applying a pre-whitening filter with the transfer function given in Eq. 5

$$H(z) = 1 + az^{-1} \qquad [5]$$

where the value for a places the corner frequency at 100 Hz.

2. **Preservation of unvoiced speech.** If the speech is delayed more than .5 to 1 ms., the high-frequency portions become decorrelated and hence are removed along with the decorrelated noise. Therefore the amount of delay, Δ, allowed for decorrelation in speech is reduced. Typical values reported for the delay in **Figure 4** were on the order of 3 to 5 samples with a sample rate of 14 kHz. This will work well as long as the noise interference is not still correlated for this short delay.

3. **Lossy LMS Algorithm.** A modified weight update equation as proposed by Gitlin (15) and given in Equation [6] was employed.

$$w_p(k+1) = w_p(k)(1-\rho) + 2\mu\epsilon(k)n_{ref}(k-p) \qquad [6]$$

The purpose of this modification is to introduce a "leak" factor so that the filter coefficient values are not "remembered" during silent portions of speech. Instead the weights are "forgotten" with a time constant set by the leak factor $(1-\rho)$. The leak rate is proportional to ρ and may be adjusted for listener preference.

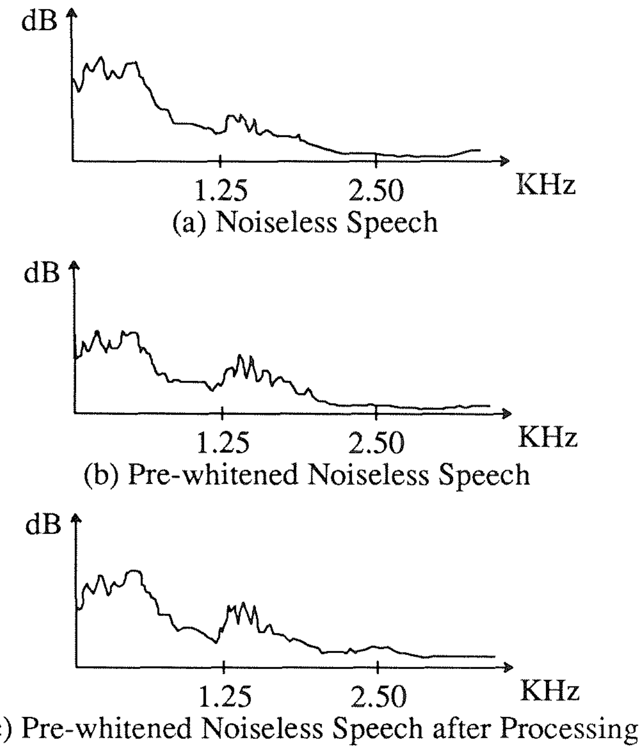

Figure 5.
Pre-whitened noiseless speech before and after processing by an adaptive LMS filter.

The effect of these techniques in preserving the high frequency information is shown in **Figure 5c.** Other techniques to accomplish similar effects have been proposed by Graupe (16). Application of another class of adaptive algorithms known as the least-squares algorithms (previously discussed) also promises to solve the problems, introduced by the LMS adaptive filter, that arise primarily due to the nonuniformity of the signal spectrum and the slow adaptation time.

Spectral Subtraction

Spectral subtraction is a technique that exploits the idea that the human hearing system is insensitive to phase information in monaural hearing applications. Boll (3) and Lim et al. (23) proposed that the short-time spectral magnitude be used to estimate the speech spectrum. While a number of methods exist to extract this estimate of the noise spectrum and subtract it from the contaminated speech signal, the basic technique uses an estimate derived from Equation [7].

$$|S(\omega)|^2 = |D(\omega)|^2 - |\hat{N}_{pri}(\omega)|^2 \qquad [7]$$

where the capital letters refer to the Fourier transforms of the $s(k)$, $d(k)$, and $n_{pri}(k)$, respectively. In Equation [7], the value of $S(\omega)$ is set to zero if the estimate of the noise, $\hat{N}_{pri}(\omega)$, is larger than $D(\omega)$.

Error Criterion

Common to all of the above techniques is the underlying notion that an increase in signal-to-noise ratio will result in improved intelligibility. Indeed, the minimization of mean square error focuses upon portions of the speech spectrum that have the most energy, and tends to lightly address the higher frequency portions of the speech spectrum that carry significant portions of the information required for speech intelligiblity. With this question is the companion inquiry, "Does an increase in signal-to-noise ratio, which does little to aid normal-hearing listeners, offer relief for hearing-impaired listeners?" Can these algorithms, which provide their best performance at higher signal-to-noise ratios (i.e., such as + 6 dB), provide hope for hearing-impaired listeners to function with processing in such an environment where they might typically require perhaps an additional 12 dB of signal to function?

To answer this question, considerable effort is planned with normal and hearing-impaired populations. However, in the midst of these efforts one is tempted to revisit the basic notions of speech intelligiblity and the human hearing system. Current efforts in noise suppression are focusing upon models of the hearing system. Examination of the Fletcher-Munson curves, which describe the contours of equal loudness, suggest the use of homomorphic digital signal processing techniques where one minimizes the error, not in the incoming acoustic pressure field, but rather in a transformed space that represents the response of the ear. Further, especially in the context of the hearing-impaired listener, that nonlinear response of the ear to a linear increase in acoustic intensity known as recruitment must be accommodated in such a model.

While several algorithms may be proposed to facilitate this modified error criterion, the frequency-domain techniques appear to come to the forefront in noise suppression. The frequency domain algorithms discussed earlier may be adapted to a modified linear error criterion by the selection of an appropriate vector μ; however, care must be taken to avoid circular convolution effects. Implementation of recruitment in such algorithms is difficult. The frequency domain technique that seems to offer the greatest flexibility is one proposed by Ferrara in 1985 (13). This implementation utilizes an FFT-based implementation with a bank of band-pass filters and a basebanded output to accomplish adaptive LMS filtering. It would appear that such algorithms offer some promise of noise suppression with intelligibility gains—but then only at positive signal-to-noise ratios.

CONCLUSION

Several implementations of techniques for accomplishing adaptive noise cancelling and noise suppression have been discussed. Application of two-channel noise cancellation has yielded significant gains in speech intelligibility, in some cases by 40 percent, while simultaneously improving signal-to-noise ratio for speech corrupted with a variety of types of corrupting noise—including speech babble and broadband noise. Reports in the literature indicate increases in signal-to-noise ratio of greater than 18 dB and of up 60 dB in specific applications. The effective application of single-channel adaptive noise suppression has shown progress with uniform reports of an increase in "quality." No data indicating intelligibility improvement for normal-hearing populations is presently available in the literature. Tests with hearing-impaired populations to determine the efficacy of noise suppression should be available to the community shortly; likewise, new algorithms which modify the error criteria in the context of the human hearing system would seem to provide the greatest hope for improving the processing of acoustic signals in the future.

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APPENDIX

The Frequency Domain Algorithm

The classical adaptive noise-cancelling problem is formulated in **Figure A1.** Defining the primary input as $d(n)$ and the reference inputs as $x_m(n)$, with n the sample index, the desired and reference inputs

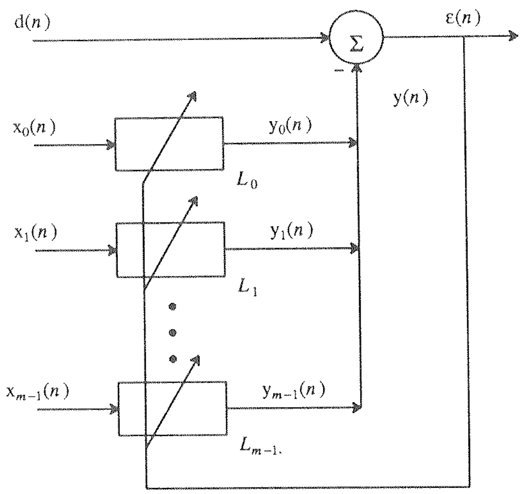

Figure A1.
Time-domain representation of a digital adaptive filter with M references of length L_m.

may be divided into blocks with index k and represented by the vectors $\mathbf{d}_L^T(k)$ and $\mathbf{x}_{L_m,m}^T(k)$ as follows

$$\mathbf{d}_L^T(k) = [d(kL)\ \ d(kL+1) \cdots d(kL+L-1)] \quad [A1]$$

$$\mathbf{x}_{L_m,m}^T(k) = [x_m(kL_m)\ \ x_m(kL_m + 1) \cdots x_m(kL_m + L_m - 1)] \quad [A2]$$

where

$m = 0,1,2,\cdots,M-1$ = reference channel number

$L_m = 2^{\alpha_m}$

$L = 2^{\alpha}$

and

α,α_m = integers specifying the block lengths.

Transforms may be obtained using the matrix $\mathbf{FFT}_L$ as

$$\mathbf{FFT}_L = \begin{bmatrix} w^0 & w^0 & w^0 & \cdot & \cdot & w^0 \\ w^0 & w^1 & w^2 & \cdot & \cdot & w^{L-1} \\ w^0 & w^2 & w^4 & \cdot & \cdot & w^{2(L-1)} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \quad [A3]$$

and $w = e^{\frac{-2\pi j}{L}}$. Further let $\chi_{2L_m,m}(k)$ represent the FFT of the $(k-1)st$ and kth consecutive blocks of the mth reference given as

$$\chi_{2L_m,m}(k) = \mathbf{FFT}_{2L_m} \begin{bmatrix} \chi_{L_m,m}(k-1) \\ \chi_{L_m,m}(k) \end{bmatrix} \quad [A4]$$

and the output of the mth filter

$$\mathbf{y}_{L_m,m}^T(k) = \text{last } L_m \text{ terms of } \mathbf{FF}_{2L_m}^{-1} [\omega_{2L_m,m}(k)\ \chi_{2L_m,m}(k)] \quad [A5]$$

where the notation **A B** denotes the element by element multiplication of the two vectors **A** and **B** which results in a vector. The sum of the outputs from all filters of various lengths, L_m, blocked to L output samples is

$$\mathbf{y}_{L,m}(k) = \begin{bmatrix} \mathbf{y}_{L_m,m}(k) \\ \mathbf{y}_{L_m,m}(k+1) \\ \mathbf{y}_{L_m,m}(k+2) \\ \cdot \\ \cdot \\ \mathbf{y}_{L_m,m}\left(k + \frac{L}{L_m} - 1\right) \end{bmatrix} \quad [A6]$$

and

$$\mathbf{y}_L(k) = \sum_{m=0}^{M-1} \mathbf{y}_{L,m}(k). \quad [A7]$$

Similarly, the error blocked to L samples becomes

$$\epsilon_L(k) = \mathbf{d}L_l(k) - \mathbf{y}_L(k) \quad [A8]$$

Padding with zeroes and transforming,

$$\mathbf{E}_{2L}(k) = \mathbf{FFT}_{2L} \begin{bmatrix} \mathbf{O}_L \\ \epsilon_L(k) \end{bmatrix} \quad [A9]$$

where the definition

$$\mathbf{O}_L^T = [0\,0\,0 \cdots 0]_L$$

will be used. The weight update equation using the

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method of steepest descents becomes

$$\begin{aligned}&\boldsymbol{\omega}^{-}_{2L_m,m}(k+1)\\&\quad=(1-\rho)\boldsymbol{\omega}^{-}_{2L_m,m}(k)+2\boldsymbol{\mu}\mathbf{E}_{2L_m}(k)\boldsymbol{\chi}^{*}_{2L_m,m}(k)\end{aligned}\quad\text{[A10]}$$

where the symbol * denotes conjugation, ρ specifies the rate of leakage, and the quantity $'\boldsymbol{\mu}$ is

$$\boldsymbol{\mu}=\begin{bmatrix}\mu_0 & 0 & \cdot & 0 & 0 & 0 & \cdot & 0\\ 0 & \mu_1 & \cdot & \cdot & \cdot & \cdot & \cdot & 0\\ 0 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & 0\\ \cdot & \cdot & \cdot & \mu_{L_m-1} & \cdot & \cdot & \cdot & 0\\ \cdot & \cdot & \cdot & \cdot & \mu_{L_m} & \cdot & \cdot & \cdot\\ \cdot & \cdot & \cdot & \cdot & \cdot & \mu_{L_m-1} & \cdot & \cdot\\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\\ 0 & 0 & \cdot & 0 & 0 & 0 & \cdot & \mu_1\end{bmatrix}\cdot\quad\text{[A11]}$$

The fact that the weights have been obtained by circular convolution is denoted by $\boldsymbol{\omega}^{-}$. To force the resultant output $\mathbf{y}_{L_m,m}(k)$ to correspond to a linear convolution, the frequency-domain weight vector is obtained as

$$\boldsymbol{\omega}_{2L_m,m}(k)=\mathbf{FFT}_{2L_m}\left[\begin{bmatrix}\mathbf{I}_{L_m} & \mathbf{O}_{L_m}\\ \mathbf{O}_{L_m} & \mathbf{O}_{L_m}\end{bmatrix}\mathbf{FFT}^{-1}_{2L_m}[\boldsymbol{\omega}^{-}_{2L_m}(k)]\right]\quad\text{[A12]}$$

where $\mathbf{I}_{L_m}$ is the $L_m \times L_m$ identity matrix. The truncation of the weight vector in [A12] ensures that the last half of a time-domain representation of the weights is identically zero. Mansour and Gray showed that this truncation was unnecessary with proper constraints on the input and that the weight vector converged to the Wiener solution; however in the general case, Equation [A12] is required.

The weight vector corresponding to the *mth* reference, $\boldsymbol{\omega}_{2L_m,m}(k)$, is updated once each L_m samples and the output vector $\mathbf{y}_{L_m,m}(k)$ is obtained from Equation [A5].

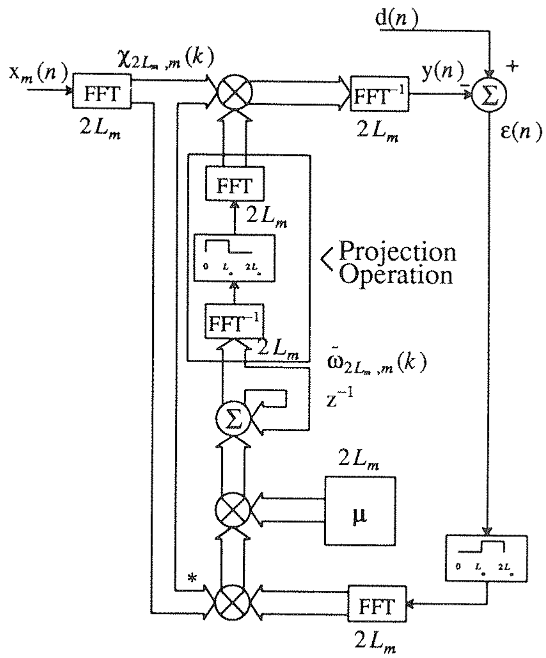

Figure A2.
Frequency-domain algorithm for the *mth* reference adaptive filter.

Figure A2 is the block diagram for the frequency-domain algorithm that is embodied in Equations [A1] through [A12].